Limit theorems for nonlinear functionals of Volterra processes via white noise analysis

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Abstract: By means of white noise analysis, we prove some limit theorems for nonlinear functionals of a given Volterra process. In particular, our results apply to fractional Brownian motion (fBm), and should be compared with the classical convergence results of the eighties by Breuer, Dobrushin, Giraitis, Major, Surgailis and Taqqu, as well as the recent advances concerning the construction of a Lvy area for fBm by Coutin, Qian and Unterberger.

1 Introduction

Fix $T > 0$, and let $B = (B_t)_{t\geqslant 0}$ be a fractional Brownian motion with Hurst index $H \in (0, 1)$, defined on some probability space $(\Omega, \mathcal{B}, P)$. Assume that $\mathcal{B}$ is the completed σ-field generated by B. Fix an integer $k \geqslant 2$ and, for $\varepsilon > 0$, consider

$$G_\varepsilon = \varepsilon^{-k(1-H)} \int_0^T h_k\big(\frac{B_{u+\varepsilon} - B_u}{\varepsilon^H}\big) du. \tag{1.1}$$

Here, and in the rest of this paper,

$$h_k(x) = (-1)^k \, e^{x^2/2} \frac{d^k}{dx^k}\big(e^{-x^2/2}\big) \tag{1.2}$$

stands for the kth Hermite polynomial. We have $h_2(x) = x^2 - 1$, $h_3(x) = x^3 - 3x$, and so on.

Since the seminal works [3, 6, 7, 19, 20] by Breuer, Dobrushin, Giraitis, Major, Surgailis and Taqqu, the three following convergence results are classical:

- If $H < 1 - \frac{1}{2k}$, then

$$\big((B_t)_{t\in[0,T]}, \varepsilon^{k(1-H)-\frac{1}{2}} G_\varepsilon\big) \xrightarrow[\varepsilon\to 0]{\text{Law}} \big((B_t)_{t\in[0,T]}, N\big), \tag{1.3}$$

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where $N \sim \mathcal{N}\big(0, T \times k! \int_0^T \rho^k(x)dx\big)$ is independent of B, with $\rho(x) = \frac{1}{2}\big(|x+1|^{2H} + |x-1|^{2H} - 2|x|^{2H}\big)$.

- If $H = 1 - \frac{1}{2k}$, then

$$\left((B_t)_{t\in[0,T]}, \frac{G_\varepsilon}{\sqrt{\log(1/\varepsilon)}}\right) \xrightarrow[\varepsilon\to 0]{\text{Law}} \big((B_t)_{t\in[0,T]}, N\big)\,, \tag{1.4}$$

  where $N \sim \mathcal{N}\big(0, T \times 2k!(1-\frac{1}{2k})^k(1-\frac{1}{k})^k\big)$ is independent of B.

- If $H > 1 - \frac{1}{2k}$, then

$$G_\varepsilon \xrightarrow[\varepsilon\to 0]{L^2(\Omega)} Z_T^{(k)}, \tag{1.5}$$

  where $Z_T^{(k)}$ denotes the Hermite random variable of order k, see Section 4.1 for its definition.

Combining (1.3) with the fact that $\sup_{0<\varepsilon\leqslant 1} E\big[|\varepsilon^{k(1-H)-\frac{1}{2}}G_\varepsilon|^p\big] < \infty$ for all $p \geqslant 1$ (use the boundedness of $\mathrm{Var}(\varepsilon^{k(1-H)-\frac{1}{2}}G_\varepsilon)$ and a classical hypercontractivity argument), we have, for all $\eta \in L^2(\Omega)$ and if $H < 1 - \frac{1}{2k}$,

$$\varepsilon^{k(1-H)-\frac{1}{2}}E[\eta\, G_\varepsilon] \xrightarrow[\varepsilon\to 0]{} E(\eta\, N) = E(\eta)E(N) = 0$$

(a similar statement holds in the critical case $H = 1 - \frac{1}{2k}$). This means that $\varepsilon^{k(1-H)-\frac{1}{2}}G_\varepsilon$ converges *weakly* in $L^2(\Omega)$ to zero. Then the following question arises. Is there a normalization of G_ε ensuring that it converges *weakly* towards a *non-zero* limit when $H \leqslant 1 - \frac{1}{2k}$? If yes, what can be said about the limit? The first purpose of the present paper is to provide an answer to this question in the framework of *white noise analysis*.

In [14], it is shown that, for all $H \in (0,1)$, the time derivative $\dot{B}$ (called the *fractional white noise*) is a distribution in the sense of Hida. We also refer to Bender [1], Biagini et al. [2] and references therein for further works on the fractional white noise.

Since we have $E(B_{u+\varepsilon} - B_u)^2 = \varepsilon^{2H}$, observe that G_ε defined in (1.1) can be rewritten as

$$G_\varepsilon = \int_0^T \left(\frac{B_{u+\varepsilon} - B_u}{\varepsilon}\right)^{\diamond k} du, \tag{1.6}$$

where $(\ldots)^{\diamond k}$ denotes the kth Wick product. In Proposition 9 below, we will show that, for all $H \in \big(\frac{1}{2} - \frac{1}{k}, 1\big)$,

$$\lim_{\varepsilon\to 0} \int_0^T \left(\frac{B_{u+\varepsilon} - B_u}{\varepsilon}\right)^{\diamond k} du = \int_0^T \dot{B}_u^{\diamond k} du, \tag{1.7}$$

where the limit is in the $(\mathcal{S})^*$ sense.

In particular, we observe two different types of asymptotic results for G_ε when $H \in \left(\frac{1}{2} - \frac{1}{k}, 1 - \frac{1}{2k}\right)$: convergence (1.7) in $(\mathcal{S})^*$ to a Hida distribution, and convergence (1.3) in law to a normal law, with rate $\varepsilon^{\frac{1}{2}-k(1-H)}$. On the other hand, when $H \in \left(1 - \frac{1}{2k}, 1\right)$ we obtain from (1.5) that the Hida distribution $\int_0^T \dot{B}_s^{\diamond k} ds$ turns out to be the square integrable random variable $Z_T^{(k)}$, which is also an interesting result by itself.

In Proposition 4 the convergence (1.7) in $(\mathcal{S})^*$ is proved for a general class of Volterra processes of the form

$$\int_0^t K(t,s) dW_s, \quad t \geqslant 0, \tag{1.8}$$

where W stands for a standard Brownian motion, provided the kernel K satisfies some suitable conditions, see Section 3.

We also provide a simple proof of the convergence (1.3) based on the recent general criterion for the convergence in distribution to a normal law of a sequence of multiple stochastic integrals established by Nualart and Peccati [15] and by Peccati and Tudor [17], which avoids the classical method of moments.

In two recent papers [9, 10], Marcus and Rosen have obtained central and non-central limit theorems for a functional of the form (1.1), where B is a mean zero Gaussian process with stationary increments such that the covariance function of B, defined by $\sigma^2(|t-s|) = \mathrm{Var}(B_t - B_s)$, is either convex (plus some additional regularity conditions), or concave, or given by $\sigma^2(h) = h^r$ with $1 < r < 2$. These theorems include the convergence (1.3), and unlike our simple proof, are based on the method of moments.

In a second part of the paper we develop a similar analysis for functionals of two independent fractional Brownian motions (or more generally, Volterra processes), related to the Lévy area. More precisely, consider two *independent* fractional Brownian motions $B^{(1)}$ and $B^{(2)}$ with (for simplicity) the same Hurst index $H \in (0,1)$. We are interested in the convergence, as $\varepsilon \to 0$, of

$$\widetilde{G}_\varepsilon := \int_0^T B_u^{(1)} \frac{B_{u+\varepsilon}^{(2)} - B_u^{(2)}}{\varepsilon} du \tag{1.9}$$

and

$$\breve{G}_\varepsilon := \int_0^T \left(\int_0^u \frac{B_{v+\varepsilon}^{(1)} - B_v^{(1)}}{\varepsilon} dv \right) \frac{B_{u+\varepsilon}^{(2)} - B_u^{(2)}}{\varepsilon} du. \tag{1.10}$$

Note that $\widetilde{G}_\varepsilon$ coincides with the ε-integral associated to the forward Russo-Vallois integral $\int_0^T B^{(1)} d^- B^{(2)}$, see e.g. [18] and references therein. In the last decade, the convergences of $\widetilde{G}_\varepsilon$ and $\breve{G}_\varepsilon$ (or of related families of random variables) have been intensively studied. Since $\varepsilon^{-1}\int_0^u (B^{(1)}_{v+\varepsilon} - B^{(1)}_v)dv$ converges pointwise to $B^{(1)}_u$ for any u, we could think that the asymptotic behaviors of $\widetilde{G}_\varepsilon$ and $\breve{G}_\varepsilon$ are very close as $\varepsilon \to 0$. Surprisingly, this is not the case. Actually, only the result for $\breve{G}_\varepsilon$ agrees with the seminal result by Coutin and Qian [4] (that is, we have convergence of $\breve{G}_\varepsilon$ in $L^2(\Omega)$ if and only if $H > 1/4$) and with the recent result by Unterberger [21] (that is, adequably renormalized, $\breve{G}_\varepsilon$ converges in law if $H < 1/4$). More precisely:

- If $H < 1/4$, then

$$\big((B^{(1)}_t, B^{(2)}_t)_{t\in[0,T]}, \varepsilon^{\frac12 - 2H}\breve{G}_\varepsilon\big) \xrightarrow[\varepsilon\to 0]{\text{Law}} \big((B^{(1)}_t, B^{(2)}_t)_{t\in[0,T]}, N\big), \quad (1.11)$$

  where $N \sim \mathscr{N}\big(0, T\breve{\sigma}^2_H\big)$ is independent of $(B^{(1)}, B^{(2)})$ and

$$\breve{\sigma}^2_H = \frac{1}{4(2H+1)(2H+2)}\int_{\mathbb{R}} \big(|x+1|^{2H} + |x-1|^{2H} - 2|x|^{2H}\big) \times \big(2|x|^{2H+2} - |x+1|^{2H+2} - |x-1|^{2H+2}\big)dx.$$

- If $H = 1/4$, then

$$\Big((B^{(1)}_t, B^{(2)}_t)_{t\in[0,T]}, \frac{\breve{G}_\varepsilon}{\sqrt{\log(1/\varepsilon)}}\Big) \xrightarrow[\varepsilon\to 0]{\text{Law}} \big((B^{(1)}_t, B^{(2)}_t)_{t\in[0,T]}, N\big), \quad (1.12)$$

  where $N \sim \mathscr{N}\big(0, T/8\big)$ is independent of $(B^{(1)}, B^{(2)})$.

- If $H > 1/4$ then

$$\breve{G}_\varepsilon \xrightarrow[\varepsilon\to 0]{L^2(\Omega)} \int_0^T B^{(1)}_u \diamond \dot{B}^{(2)}_u du = \int_0^T B^{(1)}_u dB^{(2)}_u. \quad (1.13)$$

- For all $H \in (0,1)$, we have

$$\breve{G}_\varepsilon \xrightarrow[\varepsilon\to 0]{(\mathcal{S})^*} \int_0^T B^{(1)}_u \diamond \dot{B}^{(2)}_u du. \quad (1.14)$$

But, for $\widetilde{G}_\varepsilon$, we have in contrast:

- If $H < 1/2$, then

$$\big((B^{(1)}_t, B^{(2)}_t)_{t\in[0,T]}, \varepsilon^{\frac12 - H}\widetilde{G}_\varepsilon\big) \xrightarrow[\varepsilon\to 0]{\text{Law}} \big((B^{(1)}_t, B^{(2)}_t)_{t\in[0,T]}, N \times S\big), \quad (1.15)$$

  where

$$S = \sqrt{\int_0^\infty \big(|x+1|^{2H} + |x-1|^{2H} - 2|x|^{2H}\big)dx \times \int_0^T (B^{(1)}_u)^2 du}$$

  and $N \sim \mathscr{N}(0,1)$ independent of $(B^{(1)}, B^{(2)})$.

- If $H \geqslant 1/2$ then

$$\widetilde{G}_\varepsilon \xrightarrow[\varepsilon\to 0]{L^2(\Omega)} \int_0^T B_u^{(1)} \diamond \dot{B}_u^{(2)} du = \int_0^T B_u^{(1)} dB_u^{(2)}. \tag{1.16}$$

- For all $H \in (0,1)$, we have

$$\widetilde{G}_\varepsilon \xrightarrow[\varepsilon\to 0]{(\mathcal{S})^*} \int_0^T B_u^{(1)} \diamond \dot{B}_u^{(2)} du. \tag{1.17}$$

Finally, we study the convergence, as $\varepsilon \to 0$, of the so-called ε-covariation (following the terminology of [18]) defined by

$$\widehat{G}_\varepsilon := \int_0^T \frac{B_{u+\varepsilon}^{(1)} - B_u^{(1)}}{\varepsilon} \times \frac{B_{u+\varepsilon}^{(2)} - B_u^{(2)}}{\varepsilon} du, \tag{1.18}$$

and we get:

- If $H < 3/4$, then

$$\big((B_t^{(1)}, B_t^{(2)})_{t\in[0,T]}, \varepsilon^{\frac{3}{2}-2H}\widehat{G}_\varepsilon\big) \xrightarrow[\varepsilon\to 0]{\text{Law}} \big((B_t^{(1)}, B_t^{(2)})_{t\in[0,T]}, N\big), \tag{1.19}$$

with $N \sim \mathscr{N}(0, T\widehat{\sigma}_H^2)$ independent of independent of $B^{(1)}, B^{(2)})$ and

$$\widehat{\sigma}_H^2 = \frac{1}{4}\int_{\mathbb{R}} \big(|x+1|^{2H} + |x-1|^{2H} - 2|x|^{2H}\big)^2 dx.$$

- If $H = 3/4$, then

$$\big((B_t^{(1)}, B_t^{(2)})_{t\in[0,T]}, \frac{\widehat{G}_\varepsilon}{\sqrt{\log(1/\varepsilon)}}\big) \xrightarrow[\varepsilon\to 0]{\text{Law}} \big((B_t^{(1)}, B_t^{(2)})_{t\in[0,T]}, N\big), \tag{1.20}$$

with $N \sim \mathscr{N}(0, 9T/32)$ independent of $B^{(1)}, B^{(2)})$.

- If $H > 3/4$ then

$$\widehat{G}_\varepsilon \xrightarrow[\varepsilon\to 0]{L^2(\Omega)} \int_0^T \dot{B}_u^{(1)} \diamond \dot{B}_u^{(2)} du. \tag{1.21}$$

- For all $H \in (0,1)$, we have

$$\widehat{G}_\varepsilon \xrightarrow[\varepsilon\to 0]{(\mathcal{S})^*} \int_0^T \dot{B}_u^{(1)} \diamond \dot{B}_u^{(2)} du. \tag{1.22}$$

The paper is organized as follows. In Section 2, we introduce some preliminaries on white noise analysis. Section 3 is devoted to the study, by using the language and the tools of the previous section, of the asymptotic behaviors of G_ε, $\widetilde{G}_\varepsilon$ and $\widehat{G}_\varepsilon$ in the (more general) context where B is a Volterra process. Section 4 is concerned with the fractional Brownian motion case. In Section 5 (resp. Section 6), we prove (1.3) and (1.4) (resp. (1.11), (1.12), (1.15), (1.19) and (1.20)).

2 White noise functionals

In this section, we present some preliminaries on white noise analysis. The classical point of view of the white noise distribution theory is to endow the space of tempered distributions $\mathcal{S}'(\mathbb{R})$ with a Gaussian measure $\mathbb{P}$ such that, for any rapidly decreasing function $\eta \in \mathcal{S}(\mathbb{R})$,

$$\int_{\mathcal{S}'(\mathbb{R})} e^{i\langle x,\eta\rangle}\mathbb{P}(dx) = e^{-\frac{|\eta|_0^2}{2}}.$$

Here, $|\cdot|_0$ denotes the norm in $L^2(\mathbb{R})$ and $\langle \cdot,\cdot\rangle$ the dual pairing between $\mathcal{S}'(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$. The existence of such a measure is ensured by Minlos theorem [8].

In this way we can consider the probability space $(\Omega, \mathcal{B}, \mathbb{P})$ where $\Omega = \mathcal{S}'(\mathbb{R})$. The pairing $\langle x,\xi\rangle$ can be extended, using the norm of $L^2(\Omega)$, to any function $\xi \in L^2(\mathbb{R})$. Then, $W_t = \langle \cdot, \mathbf{1}_{[0,t]}\rangle$ is a two-sided Brownian motion (by convention $\mathbf{1}_{[0,t]} = -\mathbf{1}_{[t,0]}$ if $t < 0$) and, for any $\xi \in L^2(\mathbb{R})$,

$$\langle \cdot, \xi\rangle = \int_{-\infty}^{\infty} \xi dW = I_1(\xi)$$

is the Wiener integral of ξ.

Let $\Phi \in L^2(\Omega)$. The classical Wiener chaos expansion of Φ says that there is a sequence of symmetric square integrable functions $\phi_n \in L^2(\mathbb{R}^n)$ such that

$$\Phi = \sum_{n=0}^{\infty} I_n(\phi_n), \qquad (2.23)$$

where I_n denotes the multiple stochastic integral.

2.1 The space of Hida distributions

Let us recall some basic facts about tempered distributions. Let $(\xi_n)_{n=0}^{\infty}$ be the orthonormal basis of $L^2(\mathbb{R})$ formed by the Hermite functions given by

$$\xi_n(x) = \pi^{-\frac{1}{4}}(2^n n!)^{-\frac{1}{2}} e^{-x^2/2} h_n(x), \quad x \in \mathbb{R}, \qquad (2.24)$$

for h_n the Hermite polynomials defined in (1.2). The following two facts are immediately checked: (a) there exists a constant $K_1 > 0$ such that $\|\xi_n\|_\infty \leqslant K_1(n+1)^{-1/12}$; (b) since $\xi_n' = \sqrt{\frac{n}{2}}\xi_{n-1} - \sqrt{\frac{n+1}{2}}\xi_{n+1}$, there exists a constant $K_2 > 0$ such that $\|\xi_n'\|_\infty \leqslant K_2\, n^{5/12}$.

Consider the positive self-adjoint operator A (whose inverse is Hilbert-Schmidt) given by $A = -\frac{d^2}{dx^2} + (1 + x^2)$. We have $A\xi_n = (2n + 2)\xi_n$.

For any $p \geq 0$, define the space $\mathcal{S}_p(\mathbb{R})$ as the domain of the closure of A^p. Endowed with the norm $|\xi|_p := |A^p\xi|_0$, it is a Hilbert space. Note that the norm $|\cdot|_p$ can be expressed as follows, if one uses the orthonormal basis (ξ_n)

$$|\xi|_p^2 = \sum_{n=0}^{\infty} \langle \xi, \xi_n \rangle^2 (2n+2)^{2p}.$$

We denote by $\mathcal{S}_p'(\mathbb{R})$ the dual of $\mathcal{S}_p(\mathbb{R})$. The norm in $S_p'(\mathbb{R})$ is given by (see Lemma 1.2.8 p.7 in [16])

$$|\xi|_{-p}^2 = \sum_{n=0}^{\infty} |\langle \xi, A^{-p}\xi_n \rangle|^2 = \sum_{n=0}^{\infty} \langle \xi, \xi_n \rangle^2 (2n+2)^{-2p},$$

for any $\xi \in \mathcal{S}_p'(\mathbb{R})$. One can show that the projective limit of the spaces $\mathcal{S}_p(\mathbb{R})$, $p \geqslant 0$, is $\mathcal{S}(\mathbb{R})$, and the inductive limit of the spaces $\mathcal{S}_p(\mathbb{R})'$, $p \geq 0$, is $\mathcal{S}'(\mathbb{R})$, and

$$\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$$

is a Gel'fand triple.

Now we can introduce the Gel'fand triple

$$(\mathcal{S}) \subset L^2(\Omega) \subset (\mathcal{S})^*,$$

through the second quantization operator $\Gamma(A)$. This is an unbounded and densely defined operator on $L^2(\Omega)$ given by

$$\Gamma(A)\Phi = \sum_{n=0}^{\infty} I_n(A^{\otimes n}\phi_n),$$

where Φ has the Wiener chaos expansion (2.23). If $p \geqslant 0$, we denote by $(\mathcal{S})_p$ the space of random variables $\Phi \in L^2(\Omega)$ with Wiener chaos expansion (2.23) such that

$$\|\Phi\|_p^p := E\left[|\Gamma(A)^p\Phi|^2\right] = \sum_{n=0}^{\infty} n!|\phi_n|_p^2 < \infty.$$

In the above formula $|\phi_n|_p$ denotes the norm in $\mathcal{S}_p(\mathbb{R})^{\otimes n}$. The projective limit of the spaces $(\mathcal{S})_p$, $p \geqslant 0$, is called the space of test functions and is denoted by $(\mathcal{S})$. The inductive limit of the spaces $(\mathcal{S})_{-p}$, $p \geqslant 0$, is called the space of Hida distributions and is denoted by $(\mathcal{S})^*$. The elements of $(\mathcal{S})^*$ are called Hida distributions. The main example is the time derivative of the Brownian motion defined as $\dot{W}_t = \langle \cdot, \delta_t \rangle$. One can show that that $|\delta_t|_{-p} < \infty$ for some $p > 0$.

We denote by $\ll \Phi, \Psi \gg$ the dual pairing associated with the spaces $(\mathcal{S})$ and $(\mathcal{S})^*$. On the other hand (see Theorem 3.1.6 p.36 in [16]), for any $\Phi \in (\mathcal{S})^*$ there exist $\phi_n \in \mathcal{S}(\mathbb{R}^n)'$ such that

$$\ll \Phi, \Psi \gg = \sum_{n=0}^{\infty} n!\langle \phi_n, \psi_n \rangle,$$

where $\Psi = \sum_{n=0}^{\infty} I_n(\psi_n) \in (\mathcal{S})$. Moreover there exists $p > 0$ such that:

$$\|\Phi\|_{-p}^2 = \sum_{n=0}^{\infty} n! |\phi_n|_{-p}^2.$$

Then, with a convenient abuse of notation, we say that Φ has a generalized Wiener chaos expansion of the form (2.23).

2.2 The S-transform

A useful tool to characterize elements in $(\mathcal{S})^*$ is the S-transform. The Wick exponential of a Wiener integral $I_1(\eta)$, $\eta \in L^2(\mathbb{R})$, is defined by

$$: e^{I_1(\eta)} := e^{I_1(\eta) - |\eta|_0^2/2}.$$

Then, the S-transform of an element $\Phi \in (\mathcal{S})^*$ is defined by

$$S(\Phi)(\xi) = \ll \Phi, : e^{I_1(\xi)} : \gg,$$

where $\xi \in \mathcal{S}(\mathbb{R})$. One can easily see that the S-transform is injective on $(\mathcal{S})^*$.

When $\Phi \in L^2(\Omega)$ then $S(\Phi)(\xi) = E[\Phi : e^{I_1(\xi)} :]$. For instance, the S-transform of the Wick exponential is

$$S(: e^{I_1(\eta)} :)(\xi) = e^{\langle \eta, \xi \rangle}.$$

Also, $S(W_t)(\xi) = \int_0^t \xi(s) ds$ and $S(\dot{W}_t)(\xi) = \xi(t)$.

Suppose that $\Phi \in (\mathcal{S})^*$ has a generalized Wiener chaos expansion of the form (2.23). Then, for any $\xi \in \mathcal{S}(\mathbb{R})$,

$$S(\Phi)(\xi) = \sum_{n=0}^{\infty} \langle \phi_n, \xi^{\otimes n} \rangle,$$

where the series converges absolutely (see Lemma 3.3.5 p.49 in [16]).

The Wick product of two functionals $\Psi = \sum_{n=0}^{\infty} I_n(\psi_n)$ and $\Phi = \sum_{n=0}^{\infty} I_n(\phi_n)$ belonging to $(\mathcal{S})^*$ is defined as

$$\Psi \diamond \Phi = \sum_{n,m=0}^{\infty} I_{n+m}(\psi_n \otimes \phi_m).$$

It can be proved that $\Psi \diamond \Phi \in (\mathcal{S})^*$. The following is an important property of the S-transform

$$S(\Phi \diamond \Phi)(\xi) = S(\Phi)(\xi) S(\Psi)(\xi). \tag{2.25}$$

If Ψ, Φ and $\Psi \diamond \Phi$ belong to $L^2(\Omega)$, we have $E[\Psi \diamond \Phi] = E[\Psi]E[\Phi]$.

We have the following useful characterization theorem:

Theorem 1 *A function F is the S-transform of an element $\Phi \in (\mathcal{S})^*$ if and only if the following conditions are satisfied:*

1. *For any $\xi, \eta \in \mathcal{S}$, $z \mapsto F(z\xi + \eta)$ is holomorphic on $\mathbb{C}$,*
2. *There exist non negative numbers K, a and p such that for all $\xi \in \mathcal{S}$,*

$$|F(\xi)| \leqslant K \exp(a|\xi|_p^2).$$

Proof. See Theorem 8.2 p.79 and Theorem 8.10 p.91 in [8]. ∎

In order to study the convergence of a sequence in $(\mathcal{S})^*$, we can use its S-transform thanks to the following theorem:

Theorem 2 *Let $\Phi_n \in (\mathcal{S})^*$ and $S_n = S(\Phi_n)$. Then Φ_n converges in $(\mathcal{S})^*$ if and only if the following conditions are satisfied:*

1. $\lim_{n\to\infty} S_n(\xi)$ *exists for each $\xi \in \mathcal{S}$,*
2. *There exist non negative numbers K, a and p such that for all $n \in \mathbb{N}$, $\xi \in (\mathcal{S})$,*

$$|S_n(\xi)| \leqslant K \exp(a|\xi|_p^2).$$

Proof. See Theorem 8.6 p.86 in [8]. ∎

3 Limit theorems for Volterra processes

3.1 One-dimensional case

Consider a Volterra process $B = (B_t)_{t\geqslant 0}$ of the form

$$B_t = \int_0^t K(t,s)dW_s, \tag{3.26}$$

where $K(t,s)$ satisfies $\int_0^t K(t,s)^2 ds < \infty$ for all $t > 0$, and W is the Brownian motion defined on the white noise probability space introduced in the last section. Notice that the S-transform of the random variable B_t is given by

$$S(B_t)(\xi) = \int_0^t K(t,s)\xi(s)ds, \tag{3.27}$$

for any $\xi \in \mathcal{S}(\mathbb{R})$. We introduce the following assumptions on the kernel K:

($\mathbf{H_1}$) K is continuously differentiable on $\{0 < s < t < \infty\}$ and, for any $t > 0$, we have

$$\int_0^t \left|\frac{\partial K}{\partial t}(t,s)\right|(t-s)ds < \infty;$$

($\mathbf{H_2}$) $k(t) = \int_0^t K(t,s)ds$ is continuously differentiable on $(0,\infty)$.

Consider the operator K_+ defined by

$$K_+\xi(t) = k'(t)\xi(t) + \int_0^t \frac{\partial K}{\partial t}(t,r)(\xi(r) - \xi(t))dr,$$

where $t > 0$ and $\xi \in \mathcal{S}(\mathbb{R})$. From Theorem 1, it follows that the linear mapping $\xi \to K_+\xi(t)$ is the S-transform of a Hida distribution. More precisely, according to [14], define the function

$$C(t) = |k'(t)| + \int_0^t \left|\frac{\partial K}{\partial t}(t,r)\right|(t-r)dr, \quad t \geqslant 0, \tag{3.28}$$

and observe that the following estimates hold (recall the definition (2.24) of ξ_n)

$$\begin{aligned}
|K_+\xi(t)| &\leqslant C(t)(\|\xi\|_\infty + \|\xi'\|_\infty) \\
&\leqslant C(t)\sum_{n=0}^{\infty}|\langle\xi,\xi_n\rangle|(\|\xi_n\|_\infty + \|\xi_n'\|_\infty) \\
&\leqslant C(t)M\sum_{n=0}^{\infty}|\langle\xi,\xi_n\rangle|(n+1)^{5/12} \\
&\leqslant C(t)M\sqrt{\sum_{n=0}^{\infty}|\langle\xi,\xi_n\rangle|^2(2n+2)^{17/6}}\sqrt{\sum_{n=0}^{\infty}(n+1)^{-2}} \\
&= C(t)M|\xi|_{17/12},
\end{aligned} \tag{3.29}$$

for some constants $M > 0$ whose values are not always the same from one line to another.

We have the following preliminary result.

Lemma 3 *Fix an integer $k \geqslant 1$. Let B be a Volterra process with kernel K satisfying the conditions* ($\mathbf{H_1}$) *and* ($\mathbf{H_2}$). *Assume moreover that C defined by (3.28) belongs to $L^k([0,T])$. Then the function $\xi \mapsto \int_0^T (K_+\xi(s))^k ds$ is the S-transform of an element of $(\mathcal{S})^*$. This element is denoted by $\int_0^T \dot{B}_u^{\diamond k}du$.*

Proof. We use Theorem 1. Condition (1) therein is immediately checked while, for condition (2), we just write, by using (3.29):

$$\left|\int_0^T (K_+\xi(s))^k ds\right| \leqslant \int_0^T |K_+\xi(s)|^k ds \leqslant M|\xi|_{17/12}\int_0^T C^k(s)ds.$$

■

Fix an integer $k \geq 1$, and consider the following additional condition.

($\mathbf{H_3^k}$) The maximal function $D(t) = \sup_{0<\varepsilon\leqslant\varepsilon_0} \frac{1}{\varepsilon}\int_t^{t+\varepsilon} C(s)ds$ belongs to $L^k([0,T])$ for any $T>0$, and for some $\varepsilon_0>0$.

We can now state the main result of this section.

Proposition 4 *Fix an integer $k \geqslant 1$. Let B be a Volterra process with kernel K satisfying the conditions ($\mathbf{H_1}$), ($\mathbf{H_2}$) and ($\mathbf{H_3^k}$). Then the following convergence holds:*

$$\int_0^T \left(\frac{B_{u+\varepsilon}-B_u}{\varepsilon}\right)^{\diamond k} du \xrightarrow[\varepsilon\to 0]{(\mathcal{S})^*} \int_0^T \dot{B}_u^{\diamond k} du.$$

Proof. Fix $\xi \in \mathcal{S}(\mathbb{R})$ and set

$$S_\varepsilon(\xi) = S\left(\int_0^T \left(\frac{B_{u+\varepsilon}-B_u}{\varepsilon}\right)^{\diamond k} du\right)(\xi).$$

From linearity and property (2.25) of the S-transform, we obtain

$$S_\varepsilon(\xi) = \int_0^T \frac{(S(B_{u+\varepsilon}-B_u)(\xi))^k}{\varepsilon^k} du. \tag{3.30}$$

Equation (3.27) yields

$$S(B_{u+\varepsilon}-B_u)(\xi) = \int_0^{u+\varepsilon} K(u+\varepsilon,r)\xi(r)dr - \int_0^u K(u,r)\xi(r)dr. \tag{3.31}$$

We claim that

$$\int_0^{u+\varepsilon} K(u+\varepsilon,r)\xi(r)dr - \int_0^u K(u,r)\xi(r)dr = \int_u^{u+\varepsilon} K_+\xi(s)ds. \tag{3.32}$$

Indeed, we can write

$$\begin{aligned} \int_u^{u+\varepsilon} K_+\xi(s)ds &= \int_u^{u+\varepsilon} k'(s)\xi(s)ds \\ &\quad + \int_u^{u+\varepsilon}\left(\int_0^s \frac{\partial K}{\partial s}(s,r)(\xi(r)-\xi(s))dr\right)ds \\ &= A_u^{(1)} + A_u^{(2)}. \end{aligned} \tag{3.33}$$

We have, using Fubini's theorem

$$\begin{aligned} A_u^{(2)} &= -\int_u^{u+\varepsilon} ds \int_0^s dr \frac{\partial K}{\partial s}(s,r)\int_r^s d\theta\, \xi'(\theta) \\ &= -\int_0^{u+\varepsilon} d\theta\, \xi'(\theta)\int_0^\theta dr\,(K(u+\varepsilon,r)-K(\theta\vee u,r)). \end{aligned} \tag{3.34}$$

This can be rewritten as

$$\begin{aligned} A_u^{(2)} &= -\int_0^u \left(K(u+\varepsilon,r)-K(u,r)\right)(\xi(u)-\xi(r))dr \\ &\quad -\int_u^{u+\varepsilon} d\theta \xi'(\theta)\int_0^\theta dr \left(K(u+\varepsilon,r)-K(\theta,r)\right). \end{aligned} \tag{3.35}$$

On the other hand, integration by parts yields

$$\begin{aligned} A_u^{(1)} &= \xi(u+\varepsilon)\int_0^{u+\varepsilon} K(u+\varepsilon,r)dr \qquad (3.36) \\ &\quad -\xi(u)\int_0^u K(u,r)dr - \int_u^{u+\varepsilon} ds\xi'(s)\int_0^s dr\, K(s,r). \end{aligned}$$

Therefore adding (3.36) and (3.35) yields

$$\begin{aligned} A_u^{(1)}+A_u^{(2)} &= \xi(u+\varepsilon)\int_0^{u+\varepsilon} K(u+\varepsilon,r)dr - \xi(u)\int_0^u K(u,r)dr \\ &\quad -\int_0^u \left(K(u+\varepsilon,r)-K(u,r)\right)(\xi(u)-\xi(r))dr \\ &\quad -\int_u^{u+\varepsilon} d\theta\xi'(\theta)\int_0^\theta K(u+\varepsilon,r)dr. \end{aligned} \tag{3.37}$$

Notice that, by integrating by parts:

$$\begin{aligned} -\int_u^{u+\varepsilon} d\theta\xi'(\theta)\int_0^\theta K(u+\varepsilon,r)dr &= -\xi(u+\varepsilon)\int_0^{u+\varepsilon} K(u+\varepsilon,r)dr \\ +\xi(u)\int_0^u K(u+\varepsilon,r)dr &+ \int_u^{u+\varepsilon} K(u+\varepsilon,r)\xi(r)dr. \end{aligned} \tag{3.38}$$

Thus, substituting (3.38) into (3.37) we obtain

$$A_u^{(1)}+A_u^{(2)} = \int_0^{u+\varepsilon} K(u+\varepsilon,r)\xi(r)dr - \int_0^u K(u,r)\xi(r)dr,$$

which completes the proof of (3.32). As a consequence, from (3.30)-(3.32) we obtain

$$S_\varepsilon(\xi) = \int_0^T \left(\frac{1}{\varepsilon}\int_u^{u+\varepsilon} K_+\xi(s)ds\right)^k du.$$

On the other hand, using (3.29) and the definition of the maximal function D, we get

$$\begin{aligned} \sup_{0<\varepsilon\leqslant\varepsilon_0}\left|\frac{1}{\varepsilon}\int_u^{u+\varepsilon} K_+\xi(s)ds\right|^k &\leqslant M^k|\xi|_{17/12}^k \sup_{0<\varepsilon\leqslant\varepsilon_0}\left(\frac{1}{\varepsilon}\int_u^{u+\varepsilon} C(s)ds\right)^k \\ &= M^k|\xi|_{17/12}^k D^k(u). \end{aligned} \tag{3.39}$$

Therefore, using Hypothesis ($\mathbf{H_3^k}$) and the Dominated Convergence Theorem, we have

$$\lim_{\varepsilon\to 0} S_\varepsilon(\xi) = \int_0^T (K_+\xi(s))^k ds. \tag{3.40}$$

Moreover, since $|S_\varepsilon(\xi)| \leqslant M^k |\xi|_{17/12}^k \int_0^T D^k(u)du$ for all $0 < \varepsilon \leqslant \varepsilon_0$, see (3.39), conditions (1) and (2) in Proposition 4 are fullfilled. Consequently, $\varepsilon^{-k}\int_0^T (B_{u+\varepsilon} - B_u)^{\diamond k} du$ converges in $(\mathcal{S}^*)$ as $\varepsilon \to 0$.

To finish the proof, it suffices to remark that the right-hand side of (3.40) is, by definition (see indeed Lemma 3), the S-transform of $\int_0^T \dot{B}_s^{\diamond k} ds$. ■

In [14], it is proved that, under some additional hypotheses, the mapping $t \to B_t$ is differentiable from $(0,\infty)$ to $(\mathcal{S})^*$ and that its derivative, denoted by $\dot{B}_t$, is a Hida distribution whose S-transform is $K_+\xi(t)$.

3.2 Bidimensional case

Let $W = (W_t)_{t\in\mathbb{R}}$ be a two-sided Brownian motion defined in the white noise probability space $(\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mathbb{P})$. We can consider two independent standard Brownian motions as follows: for $t \geqslant 0$, we set $W_t^{(1)} = W_t$ and $W_t^{(2)} = W_{-t}$.

In this section, we consider a bidimensional process $B = (B_t^{(1)}, B_t^{(2)})_{t\geqslant 0}$, where $B^{(1)}$ and $B^{(2)}$ are independent Volterra processes of the form

$$B_t^{(i)} = \int_0^t K(t,s)dW_s^{(i)}, \quad t \geqslant 0, \quad i = 1, 2. \tag{3.41}$$

For simplicity only, we work with the same kernel K for the two components.

First, using exactly the same lines of reasoning as in the proof of Lemma 3, we get the following result.

Lemma 5 *Let $B = (B_t^{(1)}, B_t^{(2)})_{t\geqslant 0}$ be given as above, with a kernel K satisfying the conditions ($\mathbf{H_1}$) and ($\mathbf{H_2}$). Assume morever that C defined by (3.28) belongs to $L^2([0,T])$ for any $T > 0$. Then we have the following results.*

1. *The function $\xi \mapsto \int_0^T \left(\int_0^u K_+\xi(-y)dy\right) K_+\xi(u)du$ is the S-transform of an element of $(\mathcal{S})^*$ denoted by $\int_0^T B_u^{(1)} \diamond \dot{B}_u^{(2)} du$.*
2. *The function $\xi \mapsto \int_0^T K_+\xi(-u)K_+\xi(u)du$ is the S-transform of an element of $(\mathcal{S})^*$ denoted by $\int_0^T \dot{B}_u^{(1)} \diamond \dot{B}_u^{(2)} du$.*

Now, we can state the following result.

Proposition 6 *Let $B = (B_t^{(1)}, B_t^{(2)})_{t \geqslant 0}$ be given as above, with a kernel K satisfying the conditions* $(\mathbf{H_1})$, $(\mathbf{H_2})$ *and* $(\mathbf{H_3^2})$. *Then the following convergences hold.*

$$\int_0^T B_u^{(1)} \frac{B_{u+\varepsilon}^{(2)} - B_u^{(2)}}{\varepsilon} du \quad \xrightarrow[\varepsilon \to 0]{(\mathcal{S})^*} \quad \int_0^T B_u^{(1)} \diamond \dot{B}_u^{(2)} \, du,$$

$$\int_0^T \left(\int_0^u \frac{B_{v+\varepsilon}^{(1)} - B_v^{(1)}}{\varepsilon} dv \right) \frac{B_{u+\varepsilon}^{(2)} - B_u^{(2)}}{\varepsilon} du \quad \xrightarrow[\varepsilon \to 0]{(\mathcal{S})^*} \quad \int_0^T B_u^{(1)} \diamond \dot{B}_u^{(2)} \, du,$$

$$\int_0^T \frac{B_{u+\varepsilon}^{(1)} - B_u^{(1)}}{\varepsilon} \times \frac{B_{u+\varepsilon}^{(2)} - B_u^{(2)}}{\varepsilon} du \quad \xrightarrow[\varepsilon \to 0]{(\mathcal{S})^*} \quad \int_0^T \dot{B}_u^{(1)} \diamond \dot{B}_u^{(2)} \, du.$$

Proof. Set

$$\widetilde{G}_\varepsilon = \int_0^T B_u^{(1)} \frac{B_{u+\varepsilon}^{(2)} - B_u^{(2)}}{\varepsilon} du = \int_0^T B_u^{(1)} \diamond \frac{B_{u+\varepsilon}^{(2)} - B_u^{(2)}}{\varepsilon} du.$$

From linearity and property (2.25) of the S-transform, we have

$$S(\widetilde{G}_\varepsilon)(\xi) = \frac{1}{\varepsilon} \int_0^T S(B_u^{(1)})(\xi) S(B_{u+\varepsilon}^{(2)} - B_u^{(2)})(\xi) du,$$

so that

$$S(\widetilde{G}_\varepsilon)(\xi) = \int_0^T \left(\int_0^u K_+\xi(-y) dy \right) \left(\frac{1}{\varepsilon} \int_u^{u+\varepsilon} K_+\xi(x) dx \right) du.$$

Therefore, using (3.29), (3.39), we can write

$$\begin{aligned}
|S(\widetilde{G}_\varepsilon)(\xi)| &\leqslant M^2 |\xi|_{17/12}^2 \int_0^T \left(\int_0^u C(t) dt \right) D(u) du \\
&\leqslant M^2 |\xi|_{17/12}^2 \int_0^T \left(\int_0^u D(t) dt \right) D(u) du \\
&= \frac{1}{2} M^2 |\xi|_{17/12}^2 \left(\int_0^T D(u) du \right)^2 \\
&\leqslant \frac{T}{2} M^2 |\xi|_{17/12}^2 \int_0^T D^2(u) du.
\end{aligned}$$

Hence, by the Dominated Convergence Theorem, we get

$$\lim_{\varepsilon \to 0} S(\widetilde{G}_\varepsilon)(\xi) = \int_0^T \left(\int_0^u K_+\xi(-y) dy \right) K_+\xi(u) du. \tag{3.42}$$

The right-hand side of (3.42) is the S-transform of $\int_0^T B_u^{(1)} \diamond \dot{B}_u^{(2)} du$, due to Lemma 5. Therefore, by Theorem 2, we obtain the desired result in point (1).

The proofs of the other two convergences follow exactly the same lines of reasoning, and are left to the reader. ∎

4 Fractional Brownian motion case

4.1 One-dimensional case

Consider a (one-dimensional) fractional Brownian motion (fBm) $B = (B_t)_{t\geqslant 0}$ of Hurst index $H \in (0,1)$. This means that B is a zero mean Gaussian process with covariance function

$$R_H(t,s) = E(B_t B_s) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

It is well-known that B is a Volterra process. More precisely, see [5], B has the form (3.26), with the kernel $K(t,s) = K_H(t,s)$ given by

$$K_H(t,s) = c_H \left[\left(\frac{t(t-s)}{s} \right)^{H-\frac{1}{2}} - (H-\frac{1}{2}) s^{\frac{1}{2}-H} \int_s^t u^{H-\frac{3}{2}} (u-s)^{H-\frac{1}{2}} du \right].$$

Here, c_H is a constant depending only on H. Observe that

$$\frac{\partial K_H}{\partial t}(t,s) = c_H (H-\frac{1}{2})(t-s)^{H-\frac{3}{2}} \left(\frac{s}{t}\right)^{\frac{1}{2}-H} \quad \text{for } t > s > 0. \tag{4.43}$$

Denote by $\mathscr{E}$ the set of all $\mathbb{R}$-valued step functions defined on $[0,\infty)$. Consider the Hilbert space $\mathfrak{H}$ obtained by closing $\mathscr{E}$ with respect to the inner product

$$\langle \mathbf{1}_{[0,u]}, \mathbf{1}_{[0,v]} \rangle_{\mathfrak{H}} = E(B_u B_v).$$

The mapping $\mathbf{1}_{[0,t]} \mapsto B_t$ can be extended to an isometry $\varphi \mapsto B(\varphi)$ between $\mathfrak{H}$ and the Gaussian space $\mathcal{H}_1$ associated with B. Also, write $\mathfrak{H}^{\otimes k}$ to indicate the kth tensor product of $\mathfrak{H}$. When $H > 1/2$, the inner product in the space $\mathfrak{H}$ can be written as follows, for any φ, $\psi \in \mathscr{E}$:

$$\langle \phi, \psi \rangle_{\mathfrak{H}} = H(2H-1) \int_0^\infty \int_0^\infty \phi(s)\psi(s')|s-s'|^{2H-2} ds ds'.$$

By approximation, this extends immediately to any φ, $\psi \in \mathcal{S}(\mathbb{R}) \cup \mathscr{E}$.

We will make use of the multiple integrals with respect to B (we refer to [13] for a detailed account on the properties of these integrals). For every $k \geqslant 1$, let $\mathcal{H}_k$ be the kth Wiener chaos of B, that is the closed linear subspace of $L^2(\Omega)$ generated by the random variables $\{h_k(B(\varphi)),\ \varphi \in \mathfrak{H},\ \|\varphi\|_{\mathfrak{H}} = 1\}$, where h_k is the kth Hermite polynomial (1.2). For any $k \geqslant 1$, the mapping $I_k(\varphi^{\otimes k}) = h_k(B(\varphi))$ provides a linear isometry between the *symmetric* tensor product $\mathfrak{H}^{\odot k}$ (equipped with the modified norm $\sqrt{k!}\,\|\cdot\|_{\mathfrak{H}^{\otimes k}}$) and the kth Wiener chaos $\mathcal{H}_k$.

Following [12], let us now introduce the Hermite random variable $Z_T^{(k)}$ mentioned in (1.5). Fix $T > 0$, and let $k \geqslant 1$ be an integer. The family $(\varphi_\varepsilon)_{\varepsilon>0}$, defined by

$$\varphi_\varepsilon = \varepsilon^{-k} \int_0^T \mathbf{1}_{[u,u+\varepsilon]}^{\otimes k} du, \tag{4.44}$$

satisfies

$$
\begin{aligned}
&\lim_{\varepsilon,\eta\to 0}\langle\varphi_\varepsilon,\varphi_\eta\rangle_{\mathfrak{H}^{\otimes k}}\\
= \ & H^k(2H-1)^k\int_{[0,T]^2}|s-s'|^{(2H-2)k}dsds' = c_{k,H}\,T^{(2H-2)k+2}
\end{aligned}
\tag{4.45}
$$

with $c_{k,H}=\frac{H^k(2H-1)^k}{(Hk-k+1)(2Hk-2k+1)}$. This implies that φ_ε converges, as ε tends to zero, to an element of $\mathfrak{H}^{\otimes k}$. The limit, denoted by $\pi^k_{\mathbf{1}_{[0,T]}}$, can be characterized as follows. For any $\xi_i\in\mathcal{S}(\mathbb{R})$, $i=1,\ldots,k$, we have

$$
\begin{aligned}
&\langle\pi^k_{\mathbf{1}_{[0,T]}},\xi_1\otimes\ldots\otimes\xi_k\rangle_{\mathfrak{H}^{\otimes k}}\\
= \ & \lim_{\varepsilon\to0}\langle\varphi_\varepsilon,\xi_1\otimes\ldots\otimes\xi_k\rangle_{\mathfrak{H}^{\otimes k}}\\
= \ & \lim_{\varepsilon\to0}\varepsilon^{-k}\int_0^T du\prod_{i=1}^k\langle\mathbf{1}_{[u,u+\varepsilon]},\xi_i\rangle_{\mathfrak{H}}\\
= \ & \lim_{\varepsilon\to0}\varepsilon^{-k}H^k(2H-1)^k\int_0^T du\prod_{i=1}^k\int_u^{u+\varepsilon}ds\int_0^T dr|s-r|^{2H-2}\xi_i(r)\\
= \ & H^k(2H-1)^k\int_0^T du\prod_{i=1}^k\int_0^T dr|u-r|^{2H-2}\xi_i(r).
\end{aligned}
$$

We define the kth Hermite random variable by $Z_T^{(k)}=I_k(\pi^k_{\mathbf{1}_{[0,T]}})$. Note that, by using the isometry formula for multiple integrals and since $G_\varepsilon=I_k(\varphi_\varepsilon)$, the convergence (1.5) is just a corollary of our construction of $Z_T^{(k)}$. Moreover, by (4.45), we have

$$
E\big[(Z_T^{(k)})^2\big]=c_{k,H}\times t^{(2H-2)k+2}.
$$

We will need the following preliminary result.

Lemma 7 *1. The fBm B verifies the assumptions $(\mathbf{H_1})$, $(\mathbf{H_2})$ and $(\mathbf{H_3^k})$ if and only if $H\in\left(\frac12-\frac1k,1\right)$.*

2. If $H\in\left(\frac12-\frac1k,1\right)$, then $\int_0^T\dot B_u^{\diamond k}du$ is a well-defined element of $(\mathcal{S})^*$ (in the sense of Lemma 3).

3. Assume $H>\frac12$. Then $\int_0^T\dot B_u^{\diamond k}du$ belongs to $L^2(\Omega)$ if and only if $H>1-\frac{1}{2k}$.

Proof. (1) Since

$$
k'(t)=k'_H(t)=(H+\frac12)c_1(H)t^{H-\frac12}
\tag{4.46}
$$

and

$$\int_0^t \left|\frac{\partial K_H}{\partial t}(t,s)\right|(t-s)ds = \left|\int_0^t \frac{\partial K_H}{\partial t}(t,s)(t-s)ds\right| = c_2(H)\, t^{H+\frac{1}{2}}, \quad (4.47)$$

for some constants $c_1(H)$ and $c_2(H)$, we immediately see that assumptions $(\mathbf{H_1})$ and $(\mathbf{H_2})$ are satisfied for all $H \in (0,1)$. So, it remains to focus on assumption $(\mathbf{H_3^k})$. For all $H \in (0,1)$, we have

$$\sup_{0<\varepsilon\leqslant\varepsilon_0} \frac{1}{\varepsilon}\int_t^{t+\varepsilon} s^{H-1/2}ds \leqslant t^{H-\frac{1}{2}} \vee (t+\varepsilon_0)^{H-\frac{1}{2}}, \qquad (4.48)$$

and

$$\sup_{0<\varepsilon\leqslant\varepsilon_0} \frac{1}{\varepsilon}\int_t^{t+\varepsilon} s^{H+1/2}ds \leqslant (t+\varepsilon_0)^{H+1/2}. \qquad (4.49)$$

Consequently, since $\int_0^T t^{kH-k/2}dt$ is finite when $H > \frac{1}{2} - \frac{1}{k}$, we deduce from (4.46)–(4.49) that $(\mathbf{H_3^k})$ holds in this case. Now, assume that $H \leqslant \frac{1}{2} - \frac{1}{k}$. Using the fact that $D(t) \geq C(t)$ we obtain

$$\int_0^T D^k(t)dt \geq \int_0^T C^k(t)dt = (H+\frac{1}{2})^k c_1(H)^k \int_0^T t^{kH-\frac{k}{2}}dt = \infty.$$

So, in this case, assumption $(\mathbf{H_3^k})$ is not verified.
(2) This fact is immediate to prove, combine indeed the previous point with Lemma 3.
(3) By definition of $\int_0^T \dot{B}_u^{\diamond k}du$, see Lemma 3, it is equivalent to show that the distribution $\tau^k_{\mathbf{1}_{[0,T]}}$, defined through the identity $\int_0^t \dot{B}_s^{\diamond k}ds = I_k(\tau^k_{\mathbf{1}_{[0,t]}})$, can be represented as a function belonging to $L^2([0,T]^k)$. We can write

$$\begin{aligned}\langle \tau^k_{\mathbf{1}_{[0,T]}}, \xi_1\otimes\cdots\otimes\xi_k\rangle &= \int_0^T K_+\xi_1(s)\cdots K_+\xi_k(s)ds \\ &= \int_0^T ds \prod_{i=1}^k \int_0^s \frac{\partial K_H}{\partial s}(s,r)\xi_i(r)dr,\end{aligned}$$

for any $\xi_1,\ldots,\xi_k \in \mathcal{S}(\mathbb{R})$. Observe that $K_+\xi(s) = \int_0^s \frac{\partial K_H}{\partial s}(s,r)\xi(r)dr$ because $K_H(s,s) = 0$ for $H > 1/2$. Using Fubini Theorem, we deduce that the distribution $\tau^k_{\mathbf{1}_{[0,T]}}$ can be represented as the function

$$\begin{aligned}&\tau^k_{\mathbf{1}_{[0,T]}}(x_1,\ldots,x_k) = \mathbf{1}_{[0,T]^k}(x_1,\ldots,x_k) \\ &\quad\times \int_{\max(x_1,\ldots,x_k)}^T \frac{\partial K_H}{\partial s}(s,x_1)\cdots\frac{\partial K_H}{\partial s}(s,x_k)ds.\end{aligned}$$

Then we obtain

$$\begin{aligned}
\left\|\tau^k_{\mathbf{1}_{[0,T]}}\right\|^2_{L^2([0,T]^k)} &= \int_{[0,T]^k}\int_{\max(x_1,\dots,x_k)}^T\int_{\max(x_1,\dots,x_k)}^T \frac{\partial K_H}{\partial s}(s,x_1)\cdots\frac{\partial K_H}{\partial s}(s,x_k)\\
&\quad\times\frac{\partial K_H}{\partial s}(r,x_1)\cdots\frac{\partial K_H}{\partial s}(r,x_k)dsdrdx_1\cdots dx_k\\
&= \int_{[0,T]^2}\left(\int_0^{r\wedge s}\frac{\partial K_H}{\partial s}(s,x)\frac{\partial K_H}{\partial s}(r,x)dx\right)^k drds.
\end{aligned}$$

Using the equality (4.43) and the same computations as in [13] p.278, we obtain for $s<r$,

$$\int_0^s \frac{\partial K_H}{\partial s}(s,x)\frac{\partial K_H}{\partial r}(r,x)dx = H(2H-1)(r-s)^{2H-2}. \tag{4.50}$$

Therefore

$$\|\tau^k_{\mathbf{1}_{[0,T]}}\|^2_{L^2([0,T]^k)} = (H(2H-1))^k\int_0^T\int_0^T|r-s|^{2Hk-2k}drds.$$

We immediately check that $\|\tau^k_{\mathbf{1}_{[0,T]}}\|^2_{L^2([0,T]^k)}<\infty$ if and only if $2Hk-2k>-1$, that is $H>1-\frac{1}{2k}$. Thus, in this case the Hida distribution $\int_0^T \dot B_s^{\diamond k}ds$ is a square integrable random variable with

$$E\left[\left(\int_0^T \dot B_s^{\diamond k}ds\right)^2\right] = \|\tau^k_{\mathbf{1}_{[0,T]}}\|^2_{L^2([0,T]^k)} = c_{k,H}\times T^{2Hk-2k+2}.$$

■

Remark 8 According to our result, the two distributions $\tau^k_{\mathbf{1}_{[0,T]}}$ and $\pi^k_{\mathbf{1}_{[0,T]}}$ should coincide when $H>1/2$. We can check this fact by means of elementary arguments. Let $\xi_i\in\mathcal{S}(\mathbb{R})$, $i=1,\dots,k$. From (3.32), we deduce:

$$\langle\mathbf{1}_{[u,u+\varepsilon]},\xi_i\rangle_{\mathfrak{H}} = \int_u^{u+\varepsilon}K_+\xi_i(s)ds,$$

and then

$$\lim_{\varepsilon\to0}\frac{1}{\varepsilon}\langle\mathbf{1}_{[u,u+\varepsilon]},\xi_i\rangle_{\mathfrak{H}} = K_+\xi_i(u).$$

Using (3.39) with $k=1$ for each ξ_i, and applying the Dominated Convergence Theorem since the fractional Brownian motion satisfies the assumption ($\mathbf{H_3^k}$) when $H\in\left(\frac12-\frac1k,1\right)$, we get, for φ_ε defined in (4.44):

$$\lim_{\varepsilon\to0}\langle\varphi_\varepsilon,\xi_1\otimes\ldots\otimes\xi_k\rangle_{\mathfrak{H}^{\otimes k}} = \int_0^T K_+\xi_1(u)\cdots K_+\xi_k(u)du,$$

which yields $\tau^k_{\mathbf{1}_{[0,T]}}=\pi^k_{\mathbf{1}_{[0,T]}}$.

We can now state the main result of this section.

Proposition 9 *Let $k \geqslant 2$ be an integer. If $H > \frac{1}{2} - \frac{1}{k}$ (note that this condition is immaterial for $k = 2$), the random variable*

$$G_\varepsilon = \int_0^T \left(\frac{B_{u+\varepsilon} - B_u}{\varepsilon} \right)^{\diamond k} du = \varepsilon^{-k(1-H)} \int_0^T h_k \big(\frac{B_{u+\varepsilon} - B_u}{\varepsilon^H} \big) du$$

converges in $(\mathcal{S}^*)$, as $\varepsilon \to 0$, to the Hida distribution $\int_0^T \dot{B}_u^{\diamond k} du$. Moreover, G_ε converges in $L^2(\Omega)$ if and only if $H > 1 - \frac{1}{2k}$. In this case, the limit is $\int_0^T \dot{B}_u^{\diamond k} du = Z_T^{(k)}$.

Proof. The first point follows directly from Proposition 4 and Lemma 7 (point 1). On the other hand, we already know, see (1.5), that G_ε converges in $L^2(\Omega)$ to $Z_T^{(k)}$ when $H > 1 - \frac{1}{2k}$. This implies that, when $H > 1 - \frac{1}{2k}$, $\int_0^T \dot{B}_s^{\diamond k} ds$ must be a square integrable random variable equal to $Z_T^{(k)}$. Assume now that $H \leqslant 1 - \frac{1}{2k}$. From the proof of (1.3) and (1.4) below, it follows that $E(G_\varepsilon^2)$ tends to $+\infty$ as ε tends to zero, so G_ε does not converge in $L^2(\Omega)$. ■

4.2 Bidimensional case

Let $B^{(1)}$ and $B^{(2)}$ denote two independent fractional Brownian motions of (same) Hurst index $H \in (0,1)$, defined by the stochastic integral representation (3.41) as in Section 3.2.

By combining Lemma 7 (point 1 with $k = 2$) and Lemma 5, we have the following preliminary result.

Lemma 10 *For all $H \in (0,1)$, the Hida distributions $\int_0^T B_u^{(1)} \diamond \dot{B}_u^{(2)} du$ and $\int_0^T \dot{B}_u^{(1)} \diamond \dot{B}_u^{(2)} du$ are well-defined elements of $(\mathcal{S})^*$ (in the sense of Lemma 5).*

We can now state the following result.

Proposition 11

1. *For all $H \in (0,1)$, $\widetilde{G}_\varepsilon$ defined by (1.9) converges in $(\mathcal{S}^*)$, as $\varepsilon \to 0$, to the Hida distribution $\int_0^T B_u^{(1)} \diamond \dot{B}_u^{(2)} du$. Moreover, $\widetilde{G}_\varepsilon$ converges in $L^2(\Omega)$ if and only if $H \geqslant 1/2$.*

2. *For all $H \in (0,1)$, $\breve{G}_\varepsilon$ defined by (1.10) converges in $(\mathcal{S}^*)$, as $\varepsilon \to 0$, to the Hida distribution $\int_0^T B_u^{(1)} \diamond \dot{B}_u^{(2)} du$. Moreover, $\breve{G}_\varepsilon$ converges in $L^2(\Omega)$ if and only if $H > 1/4$.*

3. *For all $H \in (0,1)$, $\hat{G}_\varepsilon$ defined by (1.18) converges in $(\mathcal{S}^*)$, as $\varepsilon \to 0$, to the Hida distribution $\int_0^T \dot{B}_u^{(1)} \diamond \dot{B}_u^{(2)} du$. Moreover, $\widehat{G}_\varepsilon$ converges in $L^2(\Omega)$ if and only if $H > 3/4$.*

Proof. (1) The first point follows directly from Proposition 6 and Lemma 7 (point 1 with $k = 2$). Assume that $H < 1/2$. From the proof of Theorem 12 below, it follows that $E(\widetilde{G}_\varepsilon^2) \to \infty$ as ε tends to zero, so $\widetilde{G}_\varepsilon$ does not converge in $L^2(\Omega)$. Assume that $H = 1/2$. By a classical result by Russo and Vallois (see e.g. the survey [18]), and since we are in this case in a martingale setting, we have that $\widetilde{G}_\varepsilon$ converges in $L^2(\Omega)$ to the Itô's integral $\int_0^T B_u^{(1)} dB_u^{(2)}$. Finally, assume that $H > 1/2$. For $\varepsilon, \eta > 0$, we have

$$E(\widetilde{G}_\varepsilon \widetilde{G}_\eta) = \frac{1}{\varepsilon\eta} \int_{[0,T]^2} \rho_{\varepsilon,\eta}(u-u') R_H(u,u') du du',$$

where

$$\rho_{\varepsilon,\eta}(x) = \frac{1}{2}[|x+\varepsilon|^{2H} + |x-\eta|^{2H} - |x|^{2H} - |x+\varepsilon-\eta|^{2H}]. \tag{4.51}$$

Remark that, as ε and η tend to zero, the quantity $(\varepsilon\eta)^{-1}\rho_{\varepsilon,\eta}(u-u')$ converges pointwise to (and is bounded by) $H(2H-1)|u-u'|^{2H-2}$. Then, by Dominated Convergence Theorem, it follows that $E(\widetilde{G}_\varepsilon \widetilde{G}_\eta)$ converges to

$$H(2H-1)\int_{[0,T]^2} |u-u'|^{2H-2} R_H(u,u') du du'$$

as $\varepsilon, \eta \to 0$, with $\int_{[0,T]^2} |u-u'|^{2H-2}|R_H(u,u')| du du' < \infty$ since $H > 1/2$. Hence, $\widetilde{G}_\varepsilon$ converges in $L^2(\Omega)$.
(2) The first point follows directly from Proposition 6 and Lemma 7 (point 1 with $k = 2$). Assume that $H \leqslant 1/4$. From the proof of Theorem 13 below, it follows that $E(\breve{G}_\varepsilon^2) \to \infty$ as ε tends to zero, so $\breve{G}_\varepsilon$ does not converge in $L^2(\Omega)$. Assume that $H > 1/4$. For $\varepsilon, \eta > 0$, we have

$$E(\breve{G}_\varepsilon \breve{G}_\eta) = \frac{1}{\varepsilon^2\eta^2} \int_{[0,T]^2} du du' \rho_{\varepsilon,\eta}(u-u') \int_0^u ds \int_0^{u'} ds' \rho_{\varepsilon,\eta}(s-s'),$$

with $\rho_{\varepsilon,\eta}$ given by (4.51). Remark that, as ε and η tend to zero, the quantity $(\varepsilon\eta)^{-1}\rho_{\varepsilon,\eta}(u-u')$ converges pointwise to $H(2H-1)|u-u'|^{2H-2}$, whereas $(\varepsilon\eta)^{-1}\int_0^u ds \int_0^{u'} ds' \rho_{\varepsilon,\eta}(s-s')$ converges pointwise to $R_H(u,u')$. Then, it follows that $E(\breve{G}_\varepsilon \breve{G}_\eta)$ converges to

$$-\frac{H}{2}(2H-1)\int_{[0,T]^2} |u-u'|^{4H-2} du du' + H\int_0^T u^{2H}(u^{2H-1} + (T-u)^{2H-1}) du$$

as $\varepsilon, \eta \to 0$, and each integral is finite since $H > 1/4$. Hence, $\breve{G}_\varepsilon$ converges in $L^2(\Omega)$.
(3) Once again, the first point follows from Proposition 6 and Lemma 7 (point 1 with $k = 2$). Assume that $H \leqslant 3/4$. From the proof of Theorem

14 below, it follows that $E(\widehat{G}_\varepsilon^2) \to \infty$ as ε tends to zero, so $\widehat{G}_\varepsilon$ does not converge in $L^2(\Omega)$. Assume now that $H > 3/4$. For $\varepsilon, \eta > 0$, we have

$$E(\widehat{G}_\varepsilon \widehat{G}_\eta) = \frac{1}{\varepsilon^2 \eta^2} \int_{[0,T]^2} \rho_{\varepsilon,\eta}(u-u')^2 du du',$$

with $\rho_{\varepsilon,\eta}$ given by (4.51). Since the quantity $(\varepsilon\eta)^{-1}\rho_{\varepsilon,\eta}(u-u')$ converges pointwise to (and is bounded by) $H(2H-1)|u-u'|^{2H-2}$, we have, by Dominated Convergence Theorem, that $E(\widehat{G}_\varepsilon \widehat{G}_\eta)$ converges to

$$H^2(2H-1)^2 \int_{[0,T]^2} |u-u'|^{4H-4} du du'$$

as $\varepsilon, \eta \to 0$, with $\int_{[0,T]^2} |u-u'|^{4H-4} du du' < \infty$ since $H > 3/4$. Hence, $\widehat{G}_\varepsilon$ converges in $L^2(\Omega)$.

■

5 A simple proof of convergences (1.3) and (1.4)

In this section we provide a simple proof these convergences by means of a recent criterion for the weak convergence of sequences of multiple stochastic integrals established in [15] and [17]. We refer to [9] for a the proof in the case of more general Gaussian processes using different kind of tools.

Let us first recall the aforementioned criterion. We continue to use the notation introduced in Section 4.1. Also, let $\{e_i,\, i \geqslant 1\}$ denote a complete orthonormal system in $\mathfrak{H}$. Given $f \in \mathfrak{H}^{\odot k}$ and $g \in \mathfrak{H}^{\odot l}$, for every $r = 0, \ldots, k \wedge l$, the *contraction* of f and g of order r is the element of $\mathfrak{H}^{\otimes(k+l-2r)}$ defined by

$$f \otimes_r g = \sum_{i_1,\ldots,i_r=1}^{\infty} \langle f, e_{i_1} \otimes \ldots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}} \otimes \langle g, e_{i_1} \otimes \ldots \otimes e_{i_r} \rangle_{\mathfrak{H}^{\otimes r}}.$$

(Notice that $f \otimes_0 g = f \otimes g$ equals the tensor product of f and g while, for $k = l$, $f \otimes_k g = \langle f, g \rangle_{\mathfrak{H}^{\otimes k}}$.) Fix $k \geqslant 2$, and let $(F_\varepsilon)_{\varepsilon>0}$ be a family of the form $F_\varepsilon = I_k(\phi_\varepsilon)$ for some $\phi_\varepsilon \in \mathfrak{H}^{\odot k}$. Assume that the variance of F_ε converges as $\varepsilon \to 0$ (to σ^2, say). The criterion by Nualart and Peccati [15] asserts that $F_\varepsilon \xrightarrow{\text{Law}} N \sim \mathscr{N}(0, \sigma^2)$ if and only if $\|\phi_\varepsilon \otimes_r \phi_\varepsilon\|_{\mathfrak{H}^{\otimes(2k-2r)}} \to 0$ for any $r = 1, \ldots, k-1$. In this case, due to the result proved by Peccati and Tudor [17], actually we have automatically that

$$(B_{t_1}, \ldots, B_{t_k}, F_\varepsilon) \xrightarrow{\text{Law}} (B_{t_1}, \ldots, B_{t_k}, N),$$

for all $t_k > \ldots > t_1 > 0$, with $N \sim \mathscr{N}(0, \sigma^2)$ *independent* of B.

For $x \in \mathbb{R}$, set

$$\rho(x) = \frac{1}{2}\left(|x+1|^{2H} + |x-1|^{2H} - 2|x|^{2H}\right), \tag{5.52}$$

and notice that $\rho(u-v) = E\big[(B_{u+1}-B_u)(B_{v+1}-B_v)\big]$ for all $u, v \geqslant 0$, and that $\int_{\mathbb{R}} |\rho(x)|^k dx$ is finite if and only if $H < 1 - \frac{1}{2k}$ (since $\rho(x) \sim H(2H-1)|x|^{2H-2}$ as $|x| \to \infty$)

We now proceed with the proof of (1.3). The proof of (1.4) would follow similar arguments.

Proof of (1.3). Because $\varepsilon^{k(1-H)-\frac{1}{2}} G_\varepsilon$ can be expressed as a kth multiple Wiener integral we can use the criterion by Nualart and Peccati. By the scaling property of the fBm, it is actually equivalent to consider the family of random variables $(F_\varepsilon)_{\varepsilon>0}$, where

$$F_\varepsilon = \sqrt{\varepsilon} \int_0^{T/\varepsilon} h_k(B_{u+1} - B_u) du.$$

Step 1: Convergence of the variance. We can write

$$\begin{aligned} E(F_\varepsilon^2) &= \varepsilon k! \int_0^{T/\varepsilon} du \int_0^{T/\varepsilon} ds \rho(u-s)^k \\ &= \varepsilon k! \int_{-T/\varepsilon}^{T/\varepsilon} \rho(x)^k (T/\varepsilon - |x|) dx, \end{aligned}$$

where the function ρ is defined in (5.52). Therefore, by the Dominated Convergence Theorem,

$$\lim_{\varepsilon \downarrow 0} E(F_\varepsilon^2) = T k! \int_{\mathbb{R}} \rho(x)^k dx.$$

Step 2: Convergence of the contractions. Observe that the random variable $h_k(B_{u+1} - B_u)$ coincides with the multiple stochastic integral $I_k(\mathbf{1}_{[u,u+1]}^{\otimes k})$. Therefore $F_\varepsilon = I_k(\phi_\varepsilon)$, where $\phi_\varepsilon = \sqrt{\varepsilon} \int_0^{T/\varepsilon} \mathbf{1}_{[u,u+1]}^{\otimes k} du$. Let $r \in \{1, \ldots, k-1\}$. We have

$$\phi_\varepsilon \otimes_r \phi_\varepsilon = \varepsilon \int_0^{T/\varepsilon} \int_0^{T/\varepsilon} \left(\mathbf{1}_{[u,u+1]}^{\otimes(k-r)} \otimes \mathbf{1}_{[s,s+1]}^{\otimes(k-r)}\right) \rho(u-s)^r du ds.$$

As a consequence, $\|\phi_\varepsilon \otimes_r \phi_\varepsilon\|^2_{\mathfrak{H}^{\otimes(2k-2r)}}$ equals to

$$\varepsilon^2 \int_{[0,T/\varepsilon]^4} \rho(u-s)^r \rho(u'-s')^r \rho(u-u')^{k-r} \rho(s-s')^{k-r} ds ds' du du'.$$

Making the change of variables $x = u-s$, $y = u'-s'$ and $z = u-u'$, we obtain that $\|\phi_\varepsilon \otimes_r \phi_\varepsilon\|^2_{\mathfrak{H}^{\otimes(2k-2r)}}$ is less than

$$A_\varepsilon = \varepsilon \int_{D_\varepsilon} |\rho(x)|^r |\rho(y)|^r |\rho(z)|^{k-r} |\rho(y+z-x)|^{k-r} dx dy dz,$$

where $D_\varepsilon = [-T/\varepsilon, T/\varepsilon]^3$. Consider the decomposition

$$\begin{aligned}
A_\varepsilon &= \varepsilon \int_{D_\varepsilon \cap \{|x|\vee|y|\vee|z| \leqslant K\}} |\rho(x)|^r |\rho(y)|^r |\rho(z)|^{k-r} |\rho(y+z-x)|^{k-r} dxdydz \\
&\quad + \varepsilon \int_{D_\varepsilon \cap \{|x|\vee|y|\vee|z| > K\}} |\rho(x)|^r |\rho(y)|^r |\rho(z)|^{k-r} |\rho(y+z-x)|^{k-r} dxdydz \\
&= B_{\varepsilon,K} + C_{\varepsilon,K}.
\end{aligned}$$

Clearly, for any fixed $K > 0$, the term $B_{\varepsilon,K}$ tends to zero because ρ is a bounded function. On the other hand, we have

$$D_\varepsilon \cap \{|x| \vee |y| \vee |z| > K\} \subset D_{\varepsilon,K,x} \cup D_{\varepsilon,K,y} \cup D_{\varepsilon,K,z},$$

where $D_{\varepsilon,K,x} = \{|x| > K\} \cap \{|y| \leqslant T/\varepsilon\} \cap \{|z| \leqslant T/\varepsilon\}$ (and a similar definition for $D_{\varepsilon,K,y}$ and $D_{\varepsilon,K,z}$). Set

$$C_{\varepsilon,K,x} = \varepsilon \int_{D_{\varepsilon,K,x}} |\rho(x)|^r |\rho(y)|^r |\rho(z)|^{k-r} |\rho(y+z-x)|^{k-r} dxdydz.$$

By Hölder's inequality, we have

$$\begin{aligned}
C_{\varepsilon,K,x} &\leqslant \varepsilon \left(\int_{D_{\varepsilon,K,x}} |\rho(x)|^k |\rho(y)|^k dxdydz \right)^{\frac{r}{k}} \\
&\qquad \times \left(\int_{D_{\varepsilon,K,x}} |\rho(z)|^k |\rho(y+z-x)|^k dxdydz \right)^{1-\frac{r}{k}} \\
&\leqslant 2T \left(\int_{\mathbb{R}} |\rho(t)|^k dt \right)^{2-\frac{r}{k}} \left(\int_{|x|>K} |\rho(x)|^k dx \right)^{\frac{r}{k}} \underset{K\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Similarly, we prove that $C_{\varepsilon,K,y} \to 0$ and $C_{\varepsilon,K,z} \to 0$ as $K \to \infty$. Finally, it suffices to choose K large enough in order to get the desired result, that is $\|\phi_\varepsilon \otimes_r \phi_\varepsilon\|_{\mathfrak{H}^{\otimes(2k-2r)}} \to 0$ as $\varepsilon \to 0$.

Step 3: Proof of the first point. By Step 1, the family

$$\left((B_t)_{t\in[0,T]}, \varepsilon^{\frac{1}{2}-2H} G_\varepsilon \right)$$

is tight in $C([0,T]) \times \mathbb{R}$. By Step 2, we also have the convergence of the finite dimensional distributions, as a byproduct of Nualart and Peccati [15] and Peccati and Tudor [17] criterions (see indeed the preliminaries at the beginning of the section). Hence, the proof of the first point is complete. ■

6 Convergences in law for some functionals related to the Lévy area of the fractional Brownian motion

Let $B^{(1)}$ and $B^{(2)}$ denote two independent fractional Brownian motions of Hurst index $H \in (0,1)$. Recall the definition (1.9) of $\widetilde{G}_\varepsilon$:

$$\widetilde{G}_\varepsilon = \int_0^T B_u^{(1)} \frac{B_{u+\varepsilon}^{(2)} - B_u^{(2)}}{\varepsilon} du.$$

Theorem 12 *Convergence in law (1.15) holds.*

Proof. We fix $H < 1/2$. The proof is divided into several steps.

Step 1: Computing the variance of $\varepsilon^{\frac{1}{2}-H}\widetilde{G}_\varepsilon$.

By using the scaling properties of the fBm, observe first that $\varepsilon^{\frac{1}{2}-H}\widetilde{G}_\varepsilon$ has the same law as

$$\widetilde{F}_\varepsilon = \varepsilon^{1/2+H} \int_0^{T/\varepsilon} B_u^{(1)} \left(B_{u+1}^{(2)} - B_u^{(2)}\right) du. \tag{6.53}$$

For $\rho(x) = \frac{1}{2}\big(|x+1|^{2H} + |x-1|^{2H} - 2|x|^{2H}\big)$, we have

$$\begin{aligned} E(\widetilde{F}_\varepsilon^2) &= \varepsilon^{1+2H} \int_0^{T/\varepsilon} du \int_0^{T/\varepsilon} ds\, R_H(u,s)\rho(u-s) \\ &= A_\varepsilon - B_\varepsilon, \end{aligned}$$

where

$$\begin{aligned} A_\varepsilon &= \varepsilon^{1+2H} \int_0^{T/\varepsilon} du\, u^{2H} \int_0^{T/\varepsilon} ds\, \rho(u-s), \\ B_\varepsilon &= \varepsilon^{1+2H} \int_0^{T/\varepsilon} du \int_0^{u} ds\, (u-s)^{2H}\rho(u-s). \end{aligned}$$

For the term B_ε we can write

$$B_\varepsilon = \varepsilon^{2H} \int_0^{T/\varepsilon} x^{2H}\rho(x)(T-\varepsilon x)dx.$$

The integral $\int_0^\infty x^{2H}\rho(x)dx$ is convergent for $H < 1/4$, while $\int_0^{T/\varepsilon} x^{2H}\rho(x)dx$ diverges as $-\frac{1}{8}\log(1/\varepsilon)$ for $H = 1/4$ and as $H(2H-1)T^{4H-1}\varepsilon^{1-4H}$ for $1/4 < H < 1/2$. The integral $\int_0^{T/\varepsilon} x^{2H+1}\rho(x)dx$ diverges as $H(2H-1)T^{4H}\varepsilon^{-4H}$. Therefore

$$\lim_{\varepsilon\to 0} B_\varepsilon = 0.$$

For A_ε, we write

$$\begin{aligned}
A_\varepsilon &= \varepsilon^{1+2H}\int_0^{T/\varepsilon} du\, u^{2H}\int_0^{T/\varepsilon} ds\, \rho(u-s)\\
&= \varepsilon^{1+2H}\left(\int_0^{T/\varepsilon} du\, u^{2H}\int_0^{u} ds\, \rho(u-s) + \int_0^{T/\varepsilon} du\, u^{2H}\int_u^{T/\varepsilon} ds\, \rho(u-s)\right)\\
&= \frac{1}{2H+1}\int_0^{T/\varepsilon} \rho(x)\big(T^{2H+1} - (\varepsilon x)^{2H+1} + (T-\varepsilon x)^{2H+1}\big)dx.
\end{aligned}$$

Hence, by Dominated Convergence Theorem, we have

$$\lim_{\varepsilon\to 0} A_\varepsilon = \frac{2T^{2H+1}}{2H+1}\int_0^\infty \rho(x)dx,$$

so that

$$\lim_{\varepsilon\to 0} \varepsilon^{1-2H} E[\widetilde{G}_\varepsilon^2] = \lim_{\varepsilon\to 0} E[\widetilde{F}_\varepsilon^2] = \frac{2T^{2H+1}}{2H+1}\int_0^\infty \rho(x)dx,$$

Step 2: Showing the convergence in law in (1.15).
By the previous step, the distributions of the family

$$\big((B_t^{(1)}, B_t^{(2)})_{t\in[0,T]}, \varepsilon^{\frac{1}{2}-H}\widetilde{G}_\varepsilon\big)_{\varepsilon>0}$$

are tight in $C([0,T]^2)\times\mathbb{R}$, and it suffices to show the convergence of the finite dimensional distributions. We need to show that, for any $\lambda\in\mathbb{R}$, any $0<t_1\leqslant\ldots\leqslant t_k$, any $\theta_1,\ldots,\theta_k\in\mathbb{R}$ and any $\mu_1,\ldots,\mu_k\in\mathbb{R}$, we have

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0} E\big[e^{i\sum_{j=1}^k \theta_j B_{t_j}^{(1)}} e^{i\sum_{j=1}^k \mu_j B_{t_j}^{(2)}} e^{i\lambda\varepsilon^{\frac{1}{2}-H}\widetilde{G}_\varepsilon}\big]\\
&\qquad = E\left[e^{-\frac{1}{2}\mathrm{Var}\left(\sum_{j=1}^k \mu_j B_{t_j}^{(2)}\right)}\right] E\left[e^{i\sum_{j=1}^k \theta_j B_{t_j}^{(1)}} e^{-\frac{\lambda^2 S^2}{2}}\right],
\end{aligned} \tag{6.54}$$

where $S=\sqrt{2\int_0^\infty \rho(x)dx\int_0^T (B_u^{(1)})^2du}$. We can write

$$\begin{aligned}
& E\big[e^{i\sum_{j=1}^k \theta_j B_{t_j}^{(1)}} e^{i\sum_{j=1}^k \mu_j B_{t_j}^{(2)}} e^{i\lambda\varepsilon^{\frac{1}{2}-H}\widetilde{G}_\varepsilon}\big]\\
=\;& E\big[e^{i\sum_{j=1}^k \theta_j B_{t_j}^{(1)}} E[e^{i\sum_{j=1}^k \mu_j B_{t_j}^{(2)}} e^{i\lambda\varepsilon^{\frac{1}{2}-H}\widetilde{G}_\varepsilon}|B^{(1)}]\big]\\
=\;& E\left[e^{i\sum_{j=1}^k \theta_j B_{t_j}^{(1)}} e^{-\lambda\,\varepsilon^{\frac{1}{2}-H}\sum_{j=1}^k \mu_j\int_0^T B_u^{(1)} E\left(B_{t_j}^{(2)}\times\frac{B_{u+\varepsilon}^{(2)}-B_u^{(2)}}{\varepsilon}\right)du}\right.\\
&\left.\times e^{-\frac{\lambda^2}{2}\varepsilon^{1-2H}\int_{[0,T]^2} B_u^{(1)}B_v^{(1)}\rho_\varepsilon(u-v)dudv} e^{-\frac{1}{2}\mathrm{Var}\left(\sum_{j=1}^k \mu_j B_{t_j}^{(2)}\right)}\right],
\end{aligned}$$

with $\rho_\varepsilon(x) = \frac{1}{2}\big(|x+\varepsilon|^{2H} + |x-\varepsilon|^{2H} - 2|x|^{2H}\big)$. Observe that

$$\int_{[0,T]^2} B_u^{(1)} B_v^{(1)} \rho_\varepsilon(u-v)\,dudv \geqslant 0$$

because $\rho_\varepsilon(u-v) = E\big[(B_{u+\varepsilon}^{(2)} - B_u^{(2)})(B_{v+\varepsilon}^{(2)} - B_v^{(2)})\big]$ is a covariance function. Moreover, for any fixed $t \geqslant 0$, we have

$$\begin{aligned}
& \int_0^T B_u^{(1)} E\left(B_t^{(2)} \times \frac{B_{u+\varepsilon}^{(2)} - B_u^{(2)}}{\varepsilon}\right) du \\
= \; & \frac{1}{2}\int_0^T B_u^{(1)} \left(\frac{(u+\varepsilon)^{2H} - u^{2H}}{\varepsilon} + \frac{|t-u|^{2H} - |t-u-\varepsilon|^{2H}}{\varepsilon}\right) du \\
\xrightarrow[\varepsilon\to 0]{\text{a.s.}} \; & H\int_0^T B_u^{(1)} \big(u^{2H-1} - |t-u|^{2H-1}\big)du.
\end{aligned}$$

Since $H < 1/2$, this implies that

$$e^{-\lambda\,\varepsilon^{\frac{1}{2}-H}\sum_{j=1}^k \mu_j \int_0^T B_u^{(1)} E\left(B_{t_j}^{(2)} \times \frac{B_{u+\varepsilon}^{(2)} - B_u^{(2)}}{\varepsilon}\right)du} \xrightarrow[\varepsilon\to 0]{\text{a.s.}} 1.$$

Hence, to get (6.54), it suffices to show that

$$E\left[e^{i\sum_{j=1}^k \theta_j B_{t_j}^{(1)}} e^{-\frac{\lambda^2}{2}\varepsilon^{1-2H}\int_{[0,T]^2} B_u^{(1)} B_v^{(1)} \rho_\varepsilon(u-v)dudv}\right] \xrightarrow[\varepsilon\to 0]{} E\left[e^{i\sum_{j=1}^k \theta_j B_{t_j}^{(1)}} e^{-\frac{\lambda^2}{2}S^2}\right]. \tag{6.55}$$

We have

$$\begin{aligned}
C_\varepsilon \; &:= \; E\left[\exp\left(i\sum_{j=1}^k \theta_j B_{t_j}^{(1)} - \frac{\lambda^2}{2}\varepsilon^{1-2H}\int_{[0,T]^2} B_u^{(1)} B_v^{(1)} \rho_\varepsilon(u-v)dudv\right)\right] \\
&= \; E\left[\exp\left(i\sum_{j=1}^k \theta_j B_{t_j}^{(1)} - \lambda^2\varepsilon^{1-2H}\int_0^T B_u^{(1)}\left(\int_0^u B_{u-x}^{(1)}\rho_\varepsilon(x)dx\right)du\right)\right] \\
&= \; E\left[\exp\left(i\sum_{j=1}^k \theta_j B_{t_j}^{(1)} - \lambda^2\varepsilon^{1-2H}\int_0^T \rho_\varepsilon(x)\left(\int_x^T B_u^{(1)} B_{u-x}^{(1)}du\right)dx\right)\right] \\
&= \; E\left[\exp\left(i\sum_{j=1}^k \theta_j B_{t_j}^{(1)} - \lambda^2\int_0^{T/\varepsilon} \rho(x)\left(\int_{\varepsilon x}^T B_u^{(1)} B_{u-\varepsilon x}^{(1)}du\right)dx\right)\right],
\end{aligned}$$

the last inequality coming from the relation $\rho_\varepsilon(x) = \varepsilon^{2H}\rho(x/\varepsilon)$. By Dominated Convergence Theorem, we get that

$$\begin{aligned} C_\varepsilon \quad &\underset{\varepsilon\to 0}{\longrightarrow} \quad E\left[\exp\left(i\sum_{j=1}^k \theta_j B_{t_j}^{(1)} - \lambda^2 \int_0^\infty \rho(x)dx \times \int_0^T (B_u^{(1)})^2 du\right)\right] \\ &= \quad E\left[\exp\left(i\sum_{j=1}^k \theta_j B_{t_j}^{(1)} - \frac{\lambda^2}{2} S^2\right)\right], \end{aligned}$$

that is (6.55). The proof of the theorem is done. ■

Recall the definition (5.52) of ρ, and the definition of $\breve{G}_\varepsilon$:

$$\breve{G}_\varepsilon = \int_0^T \left(\int_0^u \frac{B_{v+\varepsilon}^{(1)} - B_v^{(1)}}{\varepsilon} dv\right) \frac{B_{u+\varepsilon}^{(2)} - B_u^{(2)}}{\varepsilon} du.$$

Theorem 13 *Convergences in law (1.11) and (1.12) hold.*

Proof. We only show the first convergence, the proof of the second one being very similar. By using the scaling properties of the fBm, observe first that $\varepsilon^{\frac{1}{2}-2H}\breve{G}_\varepsilon$ has the same law as

$$\breve{F}_\varepsilon = \sqrt{\varepsilon} \int_0^{T/\varepsilon} \left(\int_0^u \left(B_{v+1}^{(1)} - B_v^{(1)}\right) dv\right) \left(B_{u+1}^{(2)} - B_u^{(2)}\right) du.$$

Now, we fix $H < 1/4$ and the proof is divided into several steps.

Step 1: Computing the variance of $\breve{F}_\varepsilon$. We can write

$$E(\breve{F}_\varepsilon^2) \quad = \quad \varepsilon \int_{[0,T/\varepsilon]^2} dudu'\rho(u-u') \int_0^u dv \int_0^{u'} dv'\rho(v-v'),$$

with $\rho(x) = \frac{1}{2}\left(|x+1|^{2H} + |x-1|^{2H} - 2|x|^{2H}\right)$. We have

$$\int_0^u dv \int_0^{u'} dv'\rho(v-v') = \frac{\Psi(u-u') - \Psi(u) - \Psi(u') + 2}{2(2H+1)(2H+2)},$$

where

$$\Psi(x) = 2|x|^{2H+2} - |x+1|^{2H+2} - |x-1|^{2H+2}. \tag{6.56}$$

Consider first the contribution of the term $\Psi(u-u')$. We have

$$\lim_{\varepsilon\to 0} \varepsilon \int_{[0,T/\varepsilon]^2} \rho(u-u')\Psi(u-u')dudu' = T\int_{\mathbb{R}} \rho(x)\Psi(x)dx.$$

Notice that $\rho(x) \sim H(2H-1)|x|^{2H-2}$ and $\Psi(x) \sim -(2H+2)(2H+1)|x|^{2H}$ as $|x| \to \infty$, so that $\int_{\mathbb{R}} |\rho(x)\Psi(x)|dx < \infty$ because $H < 1/4$. On the other hand, we have

$$\varepsilon \int_{[0,T/\varepsilon]^2} \rho(u-u')\Psi(u)dudu' = \varepsilon \int_0^{T/\varepsilon} du\Psi(u) \int_{u-T/\varepsilon}^{u} dx\rho(x)$$

and this converges to zero as $\varepsilon \to 0$. Indeed, since $\rho(x) \sim H(2H-1)x^{2H-2}$ as $x \to \infty$, we have $\int_u^\infty \rho(x)dx \sim Hu^{2H-1}$ as $u \to \infty$; hence, since $\int_{\mathbb{R}} \rho(x)dx = 0$, $H < 1/4$ and $\Psi(u) \sim -(2H+2)(2H+1)u^{2H}$ as $u \to \infty$, we have

$$\lim_{u\to\infty} \Psi(u) \int_{-\infty}^{u} \rho(x)dx = -\lim_{u\to\infty} \Psi(u) \int_u^\infty \rho(x)dx = 0.$$

Also, we have

$$\lim_{\varepsilon\to 0} \varepsilon \int_{[0,T/\varepsilon]^2} \rho(u-u')dudu' = \int_{\mathbb{R}} \rho(x)dx = 0.$$

Therefore, $\lim_{\varepsilon\to 0} E(\breve{F}_\varepsilon^2) = \breve{\sigma}_H^2$.

Step 2: Showing the convergence in law (1.11). We first remark that, by Step 1, the laws of the family $\big((B_t^{(1)}, B_t^{(2)})_{t\in[0,T]}, \varepsilon^{\frac{1}{2}-2H}\breve{G}_\varepsilon\big)_{\varepsilon>0}$ are tight. Therefore, we only have to prove the convergence of the finite-dimensional laws. Moreover, by the main result of Peccati and Tudor [17], it suffices to prove that

$$\varepsilon^{\frac{1}{2}-2H}\breve{G}_\varepsilon \overset{\text{Law}}{=} \breve{F}_\varepsilon \xrightarrow{\text{Law}} \mathscr{N}(0, T\breve{\sigma}_H^2) \quad \text{as } \varepsilon \to 0. \tag{6.57}$$

We have

$$\begin{aligned} E(e^{i\lambda\breve{F}_\varepsilon}) &= E\Bigg(\exp\Bigg\{-\frac{\lambda^2\varepsilon}{2}\int_{[0,T/\varepsilon]^2}(B_{u+1}^{(2)} - B_u^{(2)})(B_{u'+1}^{(2)} - B_{u'}^{(2)}) \\ &\qquad \times \left(\int_0^u \int_0^{u'} \rho(v-v')dvdv'\right) dudu'\Bigg\}\Bigg). \end{aligned}$$

Since $\rho(v-v') = E\big[(B_{v+1}^{(1)} - B_v^{(1)})(B_{v'+1}^{(1)} - B_v^{(1)})\big]$ is a covariance function, observe that the quantity inside the exponential in the right-hand side of the previous identity is negative. Hence, since $x \mapsto \exp\big(-\frac{\lambda^2}{2}x_+\big)$ is continuous and bounded by 1 on $\mathbb{R}$, (6.57) will be a consequence of the following convergence:

$$A_\varepsilon \xrightarrow{\text{law}} T\breve{\sigma}_H^2 \quad \text{as } \varepsilon \to 0, \tag{6.58}$$

with

$$A_\varepsilon := \varepsilon \int_{[0,T/\varepsilon]^2} (B_{u+1} - B_u)(B_{u'+1} - B_{u'}) \left(\int_0^u \int_0^{u'} \rho(v-v')dvdv'\right) dudu',$$

B denoting a fractional Brownian motion of Hurst index H. The proof of (6.58) will be done by showing that the expectation (resp. the variance) of A_ε tends to $T\breve{\sigma}_H^2$ (resp. zero). By Step 1, observe that

$$E(A_\varepsilon) = E(\breve{F}_\varepsilon^2) \to T\breve{\sigma}_H^2$$

as $\varepsilon \to 0$. Now, we want to show that the variance of A_ε converges to zero. Making the change of variable $s = u\varepsilon$ and $t = u'\varepsilon$ yields

$$A_\varepsilon = \varepsilon^{-1} \int_{[0,T]^2} (B_{s/\varepsilon+1} - B_{s/\varepsilon})(B_{t/\varepsilon+1} - B_{t/\varepsilon}) \left(\int_0^{s/\varepsilon} \int_0^{t/\varepsilon} \rho(v - v')dvdv' \right) dsdt,$$

which has the same distribution as

$$\begin{aligned} C_\varepsilon &= \varepsilon^{-1-2H} \int_{[0,T]^2} (B_{s+\varepsilon} - B_s)(B_{t+\varepsilon} - B_t) \left(\int_0^{s/\varepsilon} \int_0^{t/\varepsilon} \rho(u - u')dudu' \right) dsdt \\ &= \varepsilon^{-1-2H} \int_{[0,T]^2} (B_{s+\varepsilon} - B_s)(B_{t+\varepsilon} - B_t)\Lambda_\varepsilon(s,t)dsdt, \end{aligned}$$

where $\Lambda_\varepsilon(s,t) = \int_0^{s/\varepsilon} \int_0^{t/\varepsilon} \rho(u - u')dudu'$. This can be written as

$$C_\varepsilon = \varepsilon^{-1-2H} \int_{\mathbb{R}^2} B_s B_t \Sigma_\varepsilon(s,t)dsdt,$$

where

$$\begin{aligned} \Sigma_\varepsilon(s,t) &= \mathbf{1}_{[\varepsilon,T+\varepsilon]}(s)\mathbf{1}_{[\varepsilon,T+\varepsilon]}(t)\Lambda_\varepsilon(s-\varepsilon, t-\varepsilon) - \mathbf{1}_{[0,T]}(s)\mathbf{1}_{[\varepsilon,T+\varepsilon]}(t)\Lambda_\varepsilon(s, t-\varepsilon) \\ &\quad -\mathbf{1}_{[\varepsilon,T+\varepsilon]}(s)\mathbf{1}_{[0,T]}(t)\Lambda_\varepsilon(s-\varepsilon, t) + \mathbf{1}_{[0,T]}(s)\mathbf{1}_{[0,T]}(t)\Lambda_\varepsilon(s,t). \end{aligned} \tag{6.59}$$

Moreover,

$$C_\varepsilon - E(C_\varepsilon) = \varepsilon^{-1-2H} I_2 \left(\int_{\mathbb{R}^2} \mathbf{1}_{[0,s]} \otimes \mathbf{1}_{[0,t]} \Sigma_\varepsilon(s,t)dsdt \right),$$

where I_2 is the double stochastic integral with respect to B. Therefore,

$$\begin{aligned} \mathrm{Var}(C_\varepsilon) &= 2\varepsilon^{-2-4H} \left\| \int_{\mathbb{R}^2} \mathbf{1}_{[0,s]} \otimes \mathbf{1}_{[0,t]} \Sigma_\varepsilon(s,t)dsdt \right\|_{\mathfrak{H}^{\otimes 2}}^2 \\ &= 2\varepsilon^{-2-4H} \int_{\mathbb{R}^4} R_H(s,s')R_H(t,t')\Sigma_\varepsilon(s,t)\Sigma_\varepsilon(s',t')dsdtds'dt'. \end{aligned}$$

Taking into account that the partial derivatives $\frac{\partial R_H}{\partial s}$ and $\frac{\partial R_H}{\partial t}$ are integrable, we can write

$$\begin{aligned} \mathrm{Var}(C_\varepsilon) = 2\varepsilon^{-2-4H} \int_{\mathbb{R}^4} &\left(\int_0^s \frac{\partial R_H}{\partial \sigma}(\sigma, s')d\sigma \right) \left(\int_0^{t'} \frac{\partial R_H}{\partial \tau}(t,\tau)d\tau \right) \\ &\times \Sigma_\varepsilon(s,t)\Sigma_\varepsilon(s',t')dsdtds'dt'. \end{aligned}$$

Hence, by integrating by parts, we get

$$\begin{aligned}\mathrm{Var}(C_\varepsilon) = 2\varepsilon^{-2-4H} \int_{\mathbb{R}^4} \frac{\partial R_H}{\partial s}(s,s')\frac{\partial R_H}{\partial t'}(t,t')\\ \times \left(\int_0^s \times \Sigma_\varepsilon(\sigma,t)d\sigma\right)\left(\int_0^{t'} \Sigma_\varepsilon(s',\tau)d\tau\right) dsdtds'dt'.\end{aligned}$$

From (6.59) we obtain

$$\int_0^s \Sigma_\varepsilon(\sigma,t)d\sigma = \mathbf{1}_{[0,T]}(s)\left(\mathbf{1}_{[0,\varepsilon]}(t) - \mathbf{1}_{[T,T+\varepsilon]}(t)\right)\int_{s-\varepsilon}^s \Lambda_\varepsilon(\sigma, t-\varepsilon)d\sigma.$$

In the same way

$$\int_0^{t'} \Sigma_\varepsilon(s',\tau)d\tau = \mathbf{1}_{[0,T]}(t')\left(\mathbf{1}_{[0,\varepsilon]}(s') - \mathbf{1}_{[T,T+\varepsilon]}(s')\right)\int_{t'-\varepsilon}^{t'} \Lambda_\varepsilon(s'-\varepsilon, \tau)d\tau.$$

As a consequence,

$$\begin{aligned}\mathrm{Var}(C_\varepsilon) &= 2\varepsilon^{-2-4H} \int_{\mathbb{R}^4} \frac{\partial R_H}{\partial s}(s,s')\frac{\partial R_H}{\partial t'}(t,t')\left(\int_{s-\varepsilon}^s \Lambda_\varepsilon(\sigma,t-\varepsilon)d\sigma\right)\\ &\quad \times \left(\int_{t'-\varepsilon}^{t'} \Lambda_\varepsilon(s'-\varepsilon,\tau)d\tau\right)\mathbf{1}_{[0,T]}(s)\left(\mathbf{1}_{[0,\varepsilon]}(t) - \mathbf{1}_{[T,T+\varepsilon]}(t)\right)\\ &\quad \times \mathbf{1}_{[0,T]}(t')\left(\mathbf{1}_{[0,\varepsilon]}(s') - \mathbf{1}_{[T,T+\varepsilon]}(s')\right) dsdtds'dt'\\ &= \sum_{i=1}^4 H_\varepsilon^i,\end{aligned}$$

where

$$\begin{aligned}H_\varepsilon^1 &= \int_0^T \int_0^\varepsilon \int_0^\varepsilon \int_0^T G_\varepsilon(s,t,s',t')dsdtds'dt',\\ H_\varepsilon^2 &= -\int_0^T \int_T^{T+\varepsilon} \int_0^\varepsilon \int_0^T G_\varepsilon(s,t,s',t')dsdtds'dt',\\ H_\varepsilon^3 &= -\int_0^T \int_0^\varepsilon \int_T^{T+\varepsilon} \int_0^T G_\varepsilon(s,t,s',t')dsdtds'dt',\\ H_\varepsilon^4 &= \int_0^T \int_0^{T+\varepsilon} \int_0^{T+\varepsilon} \int_0^T G_\varepsilon(s,t,s',t')dsdtds'dt',\end{aligned}$$

and

$$\begin{aligned}G_\varepsilon(s,t,s',t') = 2\varepsilon^{-2-4H}\frac{\partial R_H}{\partial s}(s,s')\frac{\partial R_H}{\partial t'}(t,t')\\ \times \left(\int_{s-\varepsilon}^s \Lambda_\varepsilon(\sigma,t-\varepsilon)d\sigma\right)\left(\int_{t'-\varepsilon}^{t'} \Lambda_\varepsilon(s'-\varepsilon,\tau)d\tau\right)\end{aligned}$$

We only consider the term H_ε^1, because the other ones can be handled in the same way. We have, with Ψ given by (6.56),

$$\Lambda_\varepsilon(s,t) = \int_0^{s/\varepsilon}\int_0^{t/\varepsilon} \rho(u-u')dudu' = \frac{\Psi(\frac{s-t}{\varepsilon}) - \Psi(\frac{s}{\varepsilon}) - \Psi(\frac{t}{\varepsilon}) + 2}{2(2H+1)(2H+2)}.$$

Notice that

$$\begin{aligned}\left|\Psi(\frac{s-t}{\varepsilon})\right| &\leq \varepsilon^{-2H-2}\left|2|s-t|^{2H+2} - |s-t+\varepsilon|^{2H+2} - |s-t-\varepsilon|^{2H+2}\right| \\ &\leqslant C\varepsilon^{-2H},\end{aligned}$$

for any $s,t \in [0,T]$. Therefore, $|\Lambda_\varepsilon(s,t)| \leqslant C\varepsilon^{-2H}$, and we obtain the following estimate

$$\left|G_\varepsilon(s,t,s',t')\right| \leqslant C\varepsilon^{-8H}\left(s^{2H-1} + |s-s'|^{2H-1}\right)\left(t'^{2H-1} + |t-t'|^{2H-1}\right).$$

As a consequence,

$$\begin{aligned}\left|H_\varepsilon^1\right| &\leqslant \int_0^T\int_0^\varepsilon\int_0^\varepsilon\int_0^T \left|G_\varepsilon(s,t,s',t')\right| dsdtds'dt' \\ &\leqslant C\varepsilon^{-8H}\int_0^T\int_0^\varepsilon\int_0^\varepsilon\int_0^T \left(s^{2H-1} + |s-s'|^{2H-1}\right) \\ &\qquad \times \left(t'^{2H-1} + |t-t'|^{2H-1}\right) dsdtds'dt' \\ &\leqslant C\varepsilon^{2-8H},\end{aligned}$$

which converges to zero because $H < \frac{1}{4}$. ■

Recall the definition (1.18) of $\widehat{G}_\varepsilon$:

$$\widehat{G}_\varepsilon = \int_0^T \frac{B_{u+\varepsilon}^{(1)} - B_u^{(1)}}{\varepsilon} \times \frac{B_{u+\varepsilon}^{(2)} - B_u^{(2)}}{\varepsilon}\, du.$$

We have the following result.

Theorem 14 *Convergences (1.19) and (1.20) hold.*

Proof. We use the same trick as in [11, Remark 1.3, point 4]. Let β and $\widetilde{\beta}$ be two independent one-dimensional fractional Brownian motions of index H. Set $B^{(1)} = (\beta+\widetilde{\beta})/\sqrt{2}$ and $B^{(2)} = (\beta-\widetilde{\beta})/\sqrt{2}$. It is easily checked that $B^{(1)}$ and $B^{(2)}$ are also two independent fractional Brownian motions of index H. Moreover, we have

$$
\begin{aligned}
& \varepsilon^{\frac{3}{2}-2H}\widehat{G}_\varepsilon \\
= \;& \frac{1}{2}\,\varepsilon^{\frac{3}{2}-2H}\int_0^T \left(\frac{\beta_{u+\varepsilon}-\beta_u}{\varepsilon}\right)^2 du - \frac{1}{2}\,\varepsilon^{\frac{3}{2}-2H}\int_0^T \left(\frac{\widetilde{\beta}_{u+\varepsilon}-\widetilde{\beta}_u}{\varepsilon}\right)^2 du \\
= \;& \frac{1}{2\sqrt{\varepsilon}}\int_0^T \left(\frac{\beta_{u+\varepsilon}-\beta_u}{\varepsilon^H}\right)^2 du - \frac{1}{2\sqrt{\varepsilon}}\int_0^T \left(\frac{\widetilde{\beta}_{u+\varepsilon}-\widetilde{\beta}_u}{\varepsilon^H}\right)^2 du \\
= \;& \frac{1}{2\sqrt{\varepsilon}}\int_0^T h_2\left(\frac{\beta_{u+\varepsilon}-\beta_u}{\varepsilon^H}\right) du - \frac{1}{2\sqrt{\varepsilon}}\int_0^T h_2\left(\frac{\widetilde{\beta}_{u+\varepsilon}-\widetilde{\beta}_u}{\varepsilon^H}\right) du
\end{aligned}
\tag{6.60}
$$

The proof of the desired convergences in law are now direct consequences of the convergence (1.3) with $k=2$, taking into account that β and $\widetilde{\beta}$ are independent. ■

Remark 15 As a byproduct of the decomposition (6.60), and taking into account (1.5) for $k=2$, we get that $\int_0^T \dot{B}_u^{(1)} \diamond \dot{B}_u^{(2)} du$ and $\big(Z_T^{(2)} - \widetilde{Z}_T^{(2)}\big)/2$ have the same law when $H > 3/4$, where $\widetilde{Z}_T^{(2)}$ stands for an independent copy of the Hermite random variable $Z_T^{(2)}$.

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